Statistics 583, Problem Set 3

Wellner; 4/12/2000

Reading: Lecture Notes, Chapter 6, pages 15-36;
Lehmann, TSH (Second Ed.), Chapter 5, pages 186 - 248, (especially pages 230-245);
Ferguson, MS, Chapter 5, Sections 5.3-5.5, pages 242-251.

Due: Wednesday, April 19, 2000.

1. Let $X \sim$ Exponential(μ) and $Y \sim$ Exponential(ν) be independent random variables: thus the joint density of (X, Y) is

$$p(x, y; \mu, \nu) = \mu\nu \exp(-\mu x - \nu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \, .$$

Find a UMP unbiased test of size $\alpha = .2$ for testing:
(a) $H_0 : \mu \le \nu + 1$ against $H_1 : \mu > \nu + 1$.
(b) $H_0 : \mu = \nu$ against $H_1 : \mu \ne \nu$.
(c) $H_0 : \mu \ge 2\nu$ against $H_1 : \mu < 2\nu$.

2. Suppose that we change problem 1 as follows: $X_1, \ldots, X_m$ are i.i.d. Exponential(μ), and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(ν).
A. Find a UMP unbiased test of the hypotheses in (b) in this case.
B. Do the methods of problem 1 work for hypotheses of the form $H : g(\mu, \nu) \le 0$ versus $K : g(\mu, \nu) > 0$ and $g_1(\mu, \nu) = a\mu + b\nu + c$ for constants a, b, c? What about $g_2(\mu, \nu) = a(\mu/\nu) - b$ or $g_3(\mu, \nu) = a\lambda\nu - b$?

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, and consider testing $H : \mu \le 0$ versus $K : \mu > 0$. Let $T_1 = \sum_{i=1}^n X_i$, $T_2 = \sum_{i=1}^n X_i^2$, and set $\underline{U} \equiv (X_1, \ldots, X_n)/\sqrt{T_2} \in R^n$.
A. Show that $|\underline{U}| = \sqrt{\sum_1^n U_i^2} = 1$, so U takes values in the unit sphere S^{n-1} in R^n. Show that when $\mu = 0$, $U \sim$ Uniform(S^{n-1}).
B. Let $\underline{d} \equiv n^{-1/2}\underline{1} = n^{-1/2}(1, \ldots, 1)$; note that it is a vector of length 1. Show that

$$\langle \underline{U}, \underline{d} \rangle = \frac{n^{-1}\sum_{i=1}^n X_i}{\sqrt{T_2/n}} \, .$$

Interpret the inequality $\langle \underline{U}, \underline{d} \rangle \le c$ as an event on the sphere S^{n-1}.
C. Find the distribution of $Y_n \equiv \langle \underline{U}, \underline{d} \rangle$ under $\mu = 0$. For fixed $\alpha \in (0, 1/2)$, determine $c = c_{n,\alpha}$ so that $P_0(Y_n > c_{n,\alpha}) = \alpha$.
D. Show that the test $\phi(T_1, T_2) = 1\{Y_n > c_\alpha\}$ is equivalent to the (one-sided) one sample $t-$test $\varphi(T_1, T_2) = 1\{\sqrt{n}\overline{X}_n/S_n > t_{n-1,\alpha}\}$ where $\overline{X}_n = T_1/n$, $S_n^2 = (T_2 - T_1^2/n)/(n-1)$, and $t_{n-1,\alpha}$ is the upper α quantile of the $t-$distribution with $n-1$ degrees of freedom.

4. A. Suppose that $X_1, \ldots, X_m$ be i.i.d. $N(\mu, \sigma^2)$ and $Y_1, \ldots, Y_n$ be i.i.d. $N(\nu, \sigma^2)$ are independent. Let $N \equiv m + n$. Consider testing $H : \mu = \nu$ versus $K : \mu \neq \nu$. Show that the usual two-sample $t-$ test, reject H if $|\tau_{m,n}| > t_{m+n-2,\alpha/2}$ where

$$\tau_{m,n} \equiv \frac{\sqrt{mn/N}(\overline{X}_m - \overline{Y_n})}{S_p}$$

and

$$S_p^2 \equiv \frac{1}{m+n-2}\left\{\sum_{i=1}^{m}(X_i - \overline{X}_m)^2 + \sum_{j=1}^{m}(Y_j - \overline{Y}_n)^2\right\},$$

is the UMP unbiased test of H versus K.

B. What happens if the variances in the two populations are different? What compromise test would you use?

C. How would you approximate the power of the tests in A and B for $\mu \neq \nu$ and if the underlying distributions are not necessarily normal?