**Statistics 583, Problem Set 1**

Wellner; 3/29/2000

**Reading:** Lecture Notes, Chapter 6, pages 1-24;
Lehmann, TSH (Second Ed.), Chapter 3, pages 68 - 133;
Ferguson, MS, Chapter 5, Sections 5.1-5.3, pages 198-224.

**Due:** Wednesday, April 5, 2000.

1. Let $X_1, \ldots, X_n$ be a sample from the normal distribution $N(\mu, \sigma^2)$.
(i) If $\sigma = \sigma_0$ (known), there exists a UMP test for testing $H : \mu \le \mu_0$ against $K : \mu > \mu_0$, which rejects when $\sum_{i=1}^n (X_i - \mu_0)$ is "too large". Show this, and determine exactly what "too large" means.
(ii) If $\mu = \mu_0$ (known), there exists a UMP test for testing $H : \sigma \le \sigma_0$ against $K : \sigma > \sigma_0$ which rejects when $\sum_{i=1}^n (X_i - \mu_0)^2$ is "too large". Show this, and determine exactly what "too large" means.
(iii) Determine the asymptotic size of the UMP tests in (i) and (ii) when the $X_i$'s fail to be normal, but are from a distribution $F$ with $E_F(X^2) < \infty$ in (i) and $E_F(X^4) < \infty$ in (ii).

2. Let $X_1, \ldots, X_n$ be a sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$.
(i) Let $X = (X_1, \ldots, X_n)$. For testing $H : \theta \le \theta_0$ against $K : \theta > \theta_0$, any test is UMP at level $\alpha$ for which $E_\theta \phi(X) = \alpha$ for $\theta \le \theta_0$ and $\phi(X) = 1$ when $X_{(n)} > \theta_0$.
(ii) For testing $H : \theta = \theta_0$ against the alternative $K : \theta \ne \theta_0$ a unique UMP test exists, and is given by
$$\phi(X) = \begin{cases} 1, & \text{if } X_{(n)} \le \theta_0 \alpha^{1/n} \text{ or } X_{(n)} > \theta_0 \\ 0, & \text{if otherwise}. \end{cases}$$

3. Let $X$ be the number of successes in $n$ independent trials with probability $p$ of success, and let $\phi$ be the UMP test
$$\phi(X) = \begin{cases} 1, & \text{if } X > C, \\ \gamma, & \text{if } X = C, \\ 0, & \text{if } X < C. \end{cases}$$

(i) For $n = 7$, $p_0 = .25$, and the levels $\alpha = 0.05$, 0.1, and 0.2, determine $C$ and $\gamma$.
(ii) If $p_0 = .3$ and $\alpha = .05$, and it is desired to have power $\beta \geq 0.9$ against $p_1 = .4$, determine the necessary sample size:
(a) by using a table of the Binomial distribution;
(b) by using the normal approximation (Lehmann notes that tables an approximations are discussed in Chapter 3 of Johnson and Kotz (1969), but they can of course be found in many other places. What is the effect of correcting for continuity in using a normal approximation here?
(iii) Use the normal approximation to determine the sample size required when $\alpha = 0.05$, $\beta = .9$, $p_0 = 0.02$ and $p_1 = 0.04$.

4. Let
$$f(x; \theta, \sigma) = \frac{1}{2\sigma} \exp(-\frac{|x - \theta|}{\sigma})$$
be the double exponential density.
(i) Show that this family has monotone likelihood ratio for the parameter $\theta$ when $\sigma$ is known.
(ii) Is this an "exponential family" of distributions? Does the monotone likelihood ratio property in (i) imply that the corresponding densities of a random sample $X_1, \ldots, X_n$ i.i.d. with density $f(\cdot; \theta, \sigma)$ have monotone likelihood ratio if $\sigma$ is known?

**References**

Johnson, N. L. and Kotz, S. (1969). *Distributions in Statistics: Discrete Distributions*, Houghton Mifflin.