Statistics 582, Problem Set 8

Wellner; 2/25/98

Reading: Chapter 5, section 5.8; Ferguson, MS, Chapter 5, pages 198 - 223.
Due: Wednesday, March 4, 1998

1. Consider testing the simple hypothesis $H_0 : X \sim P_0$ versus the simple alternative $H_1 : X \sim P_1$. Let ϕ be a test of H_0 versus H_1, and suppose a prior distribution on $\{P_0, P_1\}$ is given by $(\lambda, 1-\lambda)$ with $\lambda \in (0,1)$.
A. For losses given by l_0 and l_1 as in Corollary 5.5.4, page 12, chapter 5, find the Bayes rule for this problem. Compute the ordinary risks and the Bayes risk for the Bayes rule.
B. When $l_0 = l_1 = 1$ and $\lambda = 1/2$, express the Bayes risk you found in A in terms of the total variation distance between P_0 and P_1. Explain why this relationship makes sense intuitively.
C. Does the relationship you found in B continue to hold for other values of the losses l_0 and l_1 and prior λ?

2. Suppose that $X \sim N(\theta, \sigma^2)$, and it is "known" that $\theta \in [-\tau, \tau]$.
A. Consider the class of *affine estimators* of θ given by $d(X) = cX + d$ for $c, d \in R$. Show that the minimax affine estimator of $\theta \in [-\tau, \tau]$ is given by $d_0 = c_0 X$ where $c_0 = \tau^2/(\sigma^2 + \tau^2)$ and that the minimax risk is
$$\inf_{c,d} \sup_{\theta \in [-\tau,\tau]} E_\theta(\theta - cX - d)^2 = \sigma^2\tau^2/(\sigma^2 + \tau^2)$$
B. Suppose that $\lambda(\theta)$ is a prior distribution for θ with all its mass concentrated on $[-\tau, \tau]$. Find the posterior distribution for θ for such a prior. Where is it concentrated? What can you say about the Bayes rule with respect to this prior for squared error loss?
C. How is the minimax risk over all estimators related to the minimax risk of affine estimators which you computed in A? How is it related to the Bayes risk?
D. Suppose that a sample of i.i.d. X_i's is available so that $\overline{X}_n \sim N(\theta, \sigma^2/n)$ (i.e. we can replace σ^2 by σ^2/n) in C, and suppose that the true $\theta_0 > \tau$. Describe the behavior of the posterior distributions and Bayes rule in C as $n \to \infty$.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on R, and that $\theta \sim \Gamma(\alpha, \beta)$:
$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

A. Find the Bayes rule $d_B(\underline{X})$ for estimation of θ with squared error loss $L(\theta, a) = |\theta - a|^2$. Find the Bayes rule $d_{Bw}(\underline{X})$ for estimation of θ with weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$. Is the maximum likelihood estimator among either of these families of Bayes estimators?
B. Are the Bayes estimators d_B and d_{Bw} consistent? What are the limit distributions of d_B and d_{Bw}? Compare them with the maximum likelihood estimator.
C. Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
D. Suppose that instead of the Gamma prior distribution, θ has the Pareto(θ_0, α) distribution with density λ given by

$$\lambda(\theta) = (\frac{\alpha}{\theta_0})(\frac{\theta_0}{\theta})^{\alpha+1} 1_{(\theta_0,\infty)}(\theta);$$

here $E(\theta) = \frac{\alpha}{\alpha-1}\theta_0$ where $\alpha > 1$ and $\theta_0 > 0$ are known. What can you say about the Bayes estimator for squared error loss with this prior? For what values of θ_0 is the Bayes rule consistent?

4. **Optional bonus problem:** Ferguson, MS, problem 6.2.2, page 297.