Statistics 582, Problem Set 7

Wellner; 2/18/98

Reading: Chapter 5, sections 5.6-5.8; Ferguson, MS, Chapter 2, pages 80 - 97.
Due: Wednesday, February 25, 1998.

1. Ferguson, problem 2.11.11, page 96.

2. Suppose that $X_n \equiv X \sim \text{Multinomial}_k(n, \underline{\theta})$.
   A. Suppose that the prior distribution on θ is given by a Dirichlet distribution, Dirichlet($\underline{\alpha}$):

$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \ldots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} 1_{[\underline{\theta}: \sum \theta_i = 1]} .$$

   Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4
   B. What is the posterior distribution for θ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
   C. Find a minimax estimator d_M of $\underline{\theta}$.

3. Let P and Q be two probability measures on a measurable space $(\mathcal{X}, \mathcal{B})$. Then the Hellinger distance $d_H(P, Q)$ is defined by

$$d_H^2(P, Q) = \frac{1}{2} \int [\sqrt{p} - \sqrt{q}]^2 d\mu$$

   where $p = dP/d\mu$, $q = dQ/d\mu$, and μ is any measure dominating P and Q (e.g. $P + Q$). [Note that this differs only by a factor of 1/2 from the definition we used earlier.] Similarly, the total variation distance $d_{TV}(P, Q)$ can be defined by

$$d_{TV}(P, Q) = \frac{1}{2} \int |p - q| d\mu.$$

   A. Show that

$$d_{TV}(P, Q) = \sup_{A \in \mathcal{B}} |P(A) - Q(A)|.$$

   [Hint: Let $\delta = p - q$. Since $\int p d\mu = \int q d\mu = 1$, it follows that $\int \delta d\mu = 0$ and $\int_A \delta d\mu = -\int_{A^c} \delta d\mu$ for any set A.]
   Comment: often $d_{TV}(P, Q)$ is defined by the supremum on the right side above

(without the factor of 2); then it would be proved that this equals 1/2 times the integral ($L_1(\mu)$ distance from p to q) that I used as the definition above.
B. Show that $d_{TV}(P,Q) = 1 - \int p \wedge q d\mu$.
C. Show that

$$d_H^2(P,Q) \le d_{TV}(P,Q) \le d_H(P,Q)\{2 - d_H^2(P,Q)\}^{1/2} \le \sqrt{2} d_H(P,Q).$$

[Hint: Use B to prove the first inequality; use $|p-q| = |p^{1/2} - q^{1/2}||p^{1/2} + q^{1/2}|$ and the Cauchy - Schwarz inequality to prove the second inequality.]

4. **Optional bonus problem:** Ferguson, problem 2.11.12, page 96.