**Statistics 582, Problem Set 6**

Wellner; 2/11/98

**Reading:** Chapter 5, sections 4-7; Ferguson, MS, Chapters 1 and 2, pages 43 - 80.
**Due:** Wednesday, February 18, 1998.

1. Suppose that $X \sim P_\theta$ for $\theta \in \Theta \subset R^k$ has well-defined Fisher information matrix $I(\theta)$ for $\theta$. The *Jeffreys prior* distribution $\Lambda_J$ has density $\lambda_J(\theta) = \det(I(\theta))^{1/2}$ with respect to Lebesgue measure on $\Theta$. Note that $\Lambda_J$ may not be a finite measure, and even if $\Lambda_J$ is a finite measure, it may not have total mass 1. If a prior distribution is a finite measure, then call it a *proper prior distribution*, and correspondingly if it is not a finite measure, call it an *improper prior distribution*. If the resulting posterior distribution is a finite measure, call it a *proper posterior distribution*, and (by convention) normalize it to have total mass 1.
A. Suppose that $X \sim$ Bernoulli($\theta$). Find the Jeffrey's prior density $\lambda_J$ for $\theta$. Is $\Lambda_J$ a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of $\Theta$ starting with the Jeffrey's prior.
B. Suppose that $X \sim$ Geometric($\theta$), i.e. the number of trials until the first success in i.i.d. Bernoulli trials with probability $\theta$ of success for each trial – recall Chapter 1, section 1. Find the Jeffrey's prior density $\lambda_J$ for $\theta$. Is $\Lambda_J$ a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of $\Theta$ starting with the Jeffrey's prior. If we observe $X_1, \ldots, X_n$ i.i.d. Geometric($\theta$), so that $\sum X_i \sim$ Negative Binomial($n, \theta$) is the posterior distribution "proper" for some $n$?
C. Suppose that $X \sim$ Weibull($\theta$) with $\theta = (\alpha, \beta) \in (0,\infty) \times (0,\infty)$ as in chapters 3 and 4. Find the Jeffrey's prior density $\lambda_J$ for $\theta$. Is $\Lambda_J$ a finite measure? If it is finite, what is $\Lambda_J((0,\infty)^2)$? Find the corresponding posterior distribution of $\Theta$ starting with the Jeffrey's prior.

2. **Optional bonus problem:** A. Suppose that $X \sim F$, and let $m = F^{-1}(1/2)$, $\mu = E(X)$, $\sigma^2 = Var(X)$, and we assume that $E(X^2) < \infty$. Show that

$$|m - \mu| \le \sqrt{2\sigma^2}\,.$$

Hint: use Chebychev's inequality.

B. Let $m$ and $\mu$ be the median and mean of $F$ as in A, and let $M$ be the *mode* of $F$, assuming that it is well-defined: i.e. we suppose that $F$ has density $f$ which is strictly increasing to the left of $M$ and strictly decreasing to the right of $M$. Show that $\mu \le m \le M$ if there is an $x_0$ such that

$$f(m+x) - f(m-x) \begin{cases} \ge 0 & \text{for} \quad 0 \le x < x_0 \\ \le 0 & \text{for} \quad x_0 < x < \infty \end{cases} . \tag{0.1}$$

If the inequalities in (0.1) are reversed, then $M \le m \le \mu$. Hint: show that

$$m - \mu = \int_0^\infty \{F(m-x) + F(m+x) - 1\} dx\,.$$

C. Examine (0.1) and the conclusion for the distributions Gamma$(r, 1)$ with $r = 1, 2$.