Statistics 582, Problem Set 7

Wellner; 2/14/2018

Reading: van der Vaart, *Asymptotic Statistics*, chapter 10, pages 138 - 152.
Start Reading Chapter 6, sections 6.1 and 6.2 (through page 19);
Lehmann and Romano, Chapter 3.

Due: Wednesday, February 21, 2018.

1. Let $\underline{X} \sim N_k(\underline{\theta}, I)$ and suppose that $\underline{\theta} \sim N_k(\underline{\tau}, \Sigma)$ where Σ is non-singular. Find the posterior distribution of $\underline{\theta}$. Argue directly to show that the Bernstein-von Mises theorem holds in this case.

2. Suppose that $X \sim \text{Poisson}(\theta_0)$ for some $\theta_0 \in \Theta \equiv (0, \infty)$, and suppose that the prior distribution Λ of θ is absolutely continuous with a continuous positive density at θ_0. Verify the other hypotheses of van der Vaart's Bernstein- von Mises theorem 10.1, page 141. What does the theorem say in this case?

3. Suppose that $X_n \equiv X \sim \text{Multinomial}_k(n, \underline{\theta})$.
(a) Suppose that the prior distribution on θ is given by a Dirichlet distribution, Dirichlet$(\underline{\alpha})$:
$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} 1_{[\underline{\theta}:\ \sum \theta_i = 1]} \, .$$
Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4
(b) What is the posterior distribution for θ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
(c) Find a minimax estimator d_M of $\underline{\theta}$.

4. Find the limit distribution of the minimax estimator d_M in problem 3 (i.e. $\sqrt{n}(d_M(X_n) - \theta) \to_d$ "something" and find "something"). Is d_M a regular estimator of θ?

5. **Optional bonus problem 1.** Consider the family of densities
$$p_\alpha(x) = \frac{\sin(\pi\alpha)}{2\pi(\cosh(\alpha x) + \cos(\pi\alpha))},$$
where $\alpha \in (0, 1)$. (This family of densities was introduced in the context of binary mixed effects models by Wang and Louis (2003).)

(a) Compute $p_{1/2}(x)$ and compare it to the standard logistic density by writing the standard logistic density in terms of $\cosh(x/2)$.
(b) Show that the family of densities $\{p_\alpha\}$ is log-concave for $\alpha \in (0, 1/2]$, but that it fails to be log-concave for α in $(1/2, 1)$. Thus for each $\alpha \in (0, 1/2]$ the family $\{p_\alpha(\cdot - \theta) : \theta \in \mathbb{R}\}$ has monotone likelihood ratio.

6. **Optional bonus problem 2.** Suppose that $\underline{X}_1, \underline{X}_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^d$ with $E|\underline{X}_1|^{r \vee 1} < \infty$ for some $0 < r < 2$, and let $\underline{\mu} \equiv E(\underline{X})$. Let $D_n(r) \equiv n^{-1} \sum_{i=1}^n |\underline{X}_i - \overline{\underline{X}_n}|^r$. Show that $D_n(r) \to_{a.s.} d_r \equiv E|\underline{X} - \underline{\mu}|^r$. Can you say anything about $\sqrt{n}(D_n(r) - d_r)$?