Statistics 582, Problem Set 5

Wellner; 1/31/2018

Due: Wednesday, February 7, 2018.
Reminder 1: Make-up lecture 2: Monday, February 5; 9:30-10:20 AM, EEB 003.
Reminder 2: Midterm exam, Monday February 12.
Reading: Chapter 5, sections 7-8.

1. Consider Example 5.5.4 on pages 16 and 17 of the Chapter 5 notes.
(a) Show that the variance of $\hat{\psi}$ is given by

$$Var(\hat{\psi}_n) = \frac{1}{n}\left\{\frac{1}{B}\sum_{j=1}^{B}\frac{\theta_j}{\xi_j} - \psi(\theta)^2\right\}.$$

[Hint: use the formula $Var(Y) = EVar(Y|X) + Var[E(Y|X)]$ twice.]
(b) Use the result of (a) to show that

$$Var(\hat{\psi}_n) \le \frac{1}{n\delta}$$

under the assumption that $\xi_j \ge \delta > 0$ for all $1 \le j \le B$.
(c) What if the sampling probabilities $\xi_i = 1$ for all i: do the conclusions of Example 5.5.4 and (a) and (b) above still hold?

2. A random variable X takes on values in the set $\{1, 2, 3, 4\}$ with probability distributions $p_0(x)$ or $p_1(x)$ given in the following table.

x	1	3	3	4
$p_0(x)$	.1	.3	.4	.2
$p_1(x)$	.2	.2	.2	.4

(a) Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
(b) Find a test ϕ which minimizes the sum of risks $a + b$ where $a \equiv E_0\phi$ and $b = E_1(1-\phi)$.
(c) Compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1)$. For the product laws P_0^n and P_1^n (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. P_0 or P_1 respectively), compute $\rho(P_0^n, P_1^n)$ and $H(P_0^n, P_1^n)$ for $n = 8, 32, 128$. What does this imply about the test ϕ_n based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N_k(\theta, \Sigma)$ with Σ known. Suppose that $\theta \sim N_k(\mu, \tau^2 I)$.
(a) Find the Bayes estimator for estimating θ with squared error loss $L(\theta, a) = \|\theta - a\|^2 \equiv \sum_{j=1}^k (\theta_j - a_j)^2$.
(b) Use the result of (a) to show that $\overline{X}_n$ is a minimax estimator of θ.

4. **Optional bonus problem 1:** (a) Suppose that $X \sim F$, and let $m = F^{-1}(1/2)$, $\mu = E(X)$, $\sigma^2 = Var(X)$, and we assume that $E(X^2) < \infty$. Show that

$$|m - \mu| \le \sqrt{2\sigma^2}\,.$$

Hint: use Chebychev's inequality.
(b) Show that the inequality in (a) can be improved to $|m - \mu| \le \sigma$ by using convexity of the function $\psi(x) \equiv |m - x|$ and concavity of $\varphi(x) = \sqrt{x}$.

5. **Optional bonus problem 2:** Suppose that $X \sim P_\theta$ for $\theta \in \Theta \subset R^k$ has well-defined Fisher information matrix $I(\theta)$ for θ. The *Jeffreys prior* distribution Λ_J has density $\lambda_J(\theta) = \det(I(\theta))^{1/2}$ with respect to Lebesgue measure on Θ. Note that Λ_J may not be a finite measure, and even if Λ_J is a finite measure, it may not have total mass 1. If a prior distribution is a finite measure, then call it a *proper prior distribution*, and correspondingly if it is not a finite measure, call it an *improper prior distribution*. If the resulting posterior distribution is a finite measure, call it a *proper posterior distribution*, and (by convention) normalize it to have total mass 1. See Lehmann and Casella, TPE, pages 230, 234, 287, 305.
(a) Suppose that $X \sim$ Bernoulli(θ). Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.
(b) Suppose that $X \sim$ Poisson(θ) with $\theta \in (0, \infty)$. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. Is it ever a proper posterior distribution?
(c) Suppose that $X \sim$ Geometric(θ), i.e. the number of trials until the first success in i.i.d. Bernoulli trials with probability θ of success for each trial – recall Chapter 1, section 1. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. If we observe $X_1, \ldots, X_n$ i.i.d. Geometric(θ), so that $\sum X_i \sim$ Negative Binomial(n, θ) is the posterior distribution "proper" for some n?
(d) Suppose that $X \sim$ Weibull(θ) with $\theta = (\alpha, \beta) \in (0,\infty) \times (0, \infty)$ as in chapters 3 and 4. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty)^2)$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.