Statistics 582, Problem Set 4

Wellner; 1/24/2018

Reading: Chapter 5, sections 1-4.
Ferguson, *Mathematical Statistics*, Chapter 1.
Due: Friday, February 2, 2018.
Reminder: Make up lecture 2, 9:30 - 10:20, Monday, 5 February , EEB 003.

1. Let $\Theta = \{0,1\} = \mathbf{A}$ where $0 =$ a patient has tuberculosis, $1 =$ a patient does not have tuberculosis. Let X be the number of positive reactions to two different tuberculosis tests, so that $\mathbf{X} = \{0,1,2\}$, and suppose that X has the following distributions

x	0	1	2
$p_0(x)$	.02	.13	.85
$p_1(x)$	.70	.27	.03

If the losses are given by $L(1,1) = L(0,0) = 0$, $L(0,1) = 100$, $L(1,0) = 10$, and the prior $\lambda = (\lambda_0, \lambda_1) = (.2, .8)$, find the Bayes rule d_B and the minimax rule d_M. Plot the risk set and label the non-randomized decision rules.

2. Consider testing the simple hypothesis $H_0 : X \sim P_0$ versus the simple alternative $H_1 : X \sim P_1$. Let ϕ be a test of H_0 versus H_1, and let $a \equiv E_1(1-\phi)$, $b \equiv E_0\phi$.
(a) Find a test ϕ which minimizes $a + Db$ where D is a fixed number.
(b) When $D = 1$, relate the minimized total $a + b$ to the risk and to the total variation distance $d_{TV}(P_0, P_1)$ between P_0 and P_1 (or $\int p_0 \wedge p_1 d\mu$ for a dominating measure μ, e.g. $P_0 + P_1$).
(c) Carry the computations of (b) through in the context of problem 1 when the losses are $L(0,0) = L(1,1) = 0$, $L(0,1) = 10 = L(1,0)$, and the prior distribution is $\lambda = (\lambda_0, \lambda_1) = (.3, .7)$.

3. Let $\mathcal{X} = \{0,1\}$, $\mathcal{A} = \Theta = \{1,2\}$, and assume that the losses are given by $L(1,1) = L(2,2) = 0$, $L(1,2) = a$, $L(2,1) = b$. Suppose that the statistician can observe either X or Y where

$$p_1(1) = P_1(X=1) = 2/3, \qquad p_2(1) = P_2(X=1) = 1/2,$$
$$p_1^*(1) = P_1(Y=1) = 3/4, \qquad p_2^*(1) = P_2(Y=1) = 1/2.$$

Let $\underline{\lambda} = (\lambda, 1-\lambda)$, $\lambda \in [0,1]$ be the prior distribution over Θ.
(a) Find the Bayes risk when X is observed, and similarly for Y.
(b) In the case $a = b$, $\lambda = 1/2$, would the statistician prefer to observe X or Y?
(c) For general $a \neq b$, $\lambda \in (0,1)$ would the statistician prefer to observe X or Y?

4. Let $\Theta = \mathcal{A} = \{0, 1\}$, assume the (hypothesis testing) loss function $L(0, 0) = L(1, 1) = 0$, $L(0, 1) = L(1, 0) = 1$. Suppose that we observe the random variable X with the discrete distribution $P_\theta(X = x) = 2^{-(x+\theta)} 1_{\mathbb{Z}^+ \cap [1-\theta,\infty)}(x)$. (a) Describe the set of all non-randomized decision rules.
   (b) Plot the risk set $\mathcal{R}$ in the plane. Which non-randomized rules are admissible? Why?
   (c) Can you find a non-randomized minimax rule?
   (d) What decision rules result from a Neyman-Pearson approach?
   Hint: every number $z \in [0, 1]$ has a diadic representation of the form $\sum_{x=1}^{\infty} d(x) 2^{-x}$ where $d(x) = 0$ or 1.

5. Consider Example 1.3 as given on page 4 of the Chapter 5 notes.
   (a) Consider calculation of the Bayes rule for the prior $\lambda_1 = \lambda$, $\lambda_2 = 1 - \lambda$ with $\lambda \in [0, 1]$ via computation of the posterior risk(s) as in Theorem 5.1: in particular, compute the posterior risks

$$E\{L(\boldsymbol{\theta}, d(\cdot|X))|X = x\}$$

   for $x \in \{R, B, G\}$.
   (b) Specialize your calculation in (a) to the particular prior $\lambda = 1/2$ and thereby show that the non-randomized rule $d_2 = (0, 0, 1)$ is the Bayes rule.
   (c) Determine a prior which yields all the risk points on the the line between $(1, 3.6) = (R(1, d_7), R(2, d_7))$ and $(0, 6) = (R(1, d_8), R(2, d_8))$ as risks of Bayes rules.

6. **Optional bonus problem 1:** Suppose that $\Theta = \{\theta_1, \theta_2\}$, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, and that the loss function $L(\theta, a)$ is given by the following table:

θ/a	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

   Further suppose that $P_{\theta_j}(X = 0) = 1$ for $j = 1, 2$.
   (a) Find the decision risk set $\mathcal{R}$.
   (b) Find the decision rules that are Bayes with respect to the prior distribution $\lambda = (1, 0)$.
   (c) Show that the rule d_0 for which $R(\theta_1, d_0) = 1$ and $R(\theta_2, d_0) = 1$ is Bayes with respect to $\lambda = (1, 0)$ and also minimax, but that it is not admissible.
   (d) Relate this example to our theorem about admissibility of Bayes rules.