**Statistics 582, "Practice" Final Exam**

Wellner; 3/8/2018

1. (30 points) **Define** *any three* of the following terms. In each case, briefly provide an appropriate context for your definition.

   **(a)** A family of distributions $\mathcal{P} = \{P_\theta : \theta \in \Theta \subset \mathbb{R}\}$ (with densities $p_\theta = dP_\theta/d\mu$ with respect to a dominating measure $\mu$) with *monotone likelihood ratio.*

   **(b)** An *unbiased* level $\alpha$ test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   **(c)** A similar test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   **(d)** A *minimax decision rule* in a general decision problem with loss function $L(\theta, d)$.

2. (30 points) **State** *any three* of the following results:

   **(a)** The Neyman - Pearson lemma.

   **(b)** The Karlin-Rubin theorem.

   **(c)** A theorem relating similar tests to tests with Neyman structure.

   **(d)** A theorem about admissibility properties of the sample mean $\overline{X}$ when sampling from a normal distribution on $\mathbb{R}$ and a contrasting theorem for sampling from a normal distribution on $\mathbb{R}^d$.

   **(e)** A theorem relating Bayes rules to minimax rules and least favorable prior distributions.

   **(f)** A conditional limit theorem about the large sample behavior of posterior distributions

   **Everyone should do problems 3.**

3. (36 points) A random variable $X$ takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p_0(x)$ | .05 | .07 | .40 | .48 |
| $p_1(x)$ | .15 | .35 | .35 | .15 |

   **(a)** For the usual $0 - 1$ loss, find a most powerful test of size .10 for testing $H : p = p_0$ versus $K : p = p_1$ and determine its power.

**(b)** Find a test $\phi$ which minimizes the sum of risks $E_0\phi + E_1(1-\phi)$. What is the relationship between the minimized sum of risks and the total variation distance between $P_1$ and $P_2$ and why does this make intuitive sense?

**(c)** If the losses are $L(1,1) = L(0,0) = 0$, $L(0,1) = 3$, $L(1,0) = 2$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.4, .6)$, find the Bayes rule $d_B$ and the minimax rule $d_M$.

**Do any two of problems 4 - 6.**

4. (40 points) State and prove the "short form" of the generalized Neyman-Pearson lemma.

5. (40 points) Prove the following theorem (in the context of finitely many states of nature): Suppose that $d_0$ is Bayes with respect to $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\lambda_i > 0$ for $i = 1, \ldots, l$. Then $d_0$ is admissible.

6. (40 points) In the context of $X_1, \ldots, X_n$ i.i.d. either $P$ or $Q$ with densities $p$ and $q$ with respect to some dominating measure $\mu$, show that Neyman - Pearson tests are both size and power consistent under a simple restriction on the constants specifying the tests.

**Do either problem 7 or problem 8.**

7. (36 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson($\theta$) random variables so that $P(X_1 = x) = \exp(-\theta)\theta^x/x!$ for $x \in \{0, 1, \ldots\}$ and $\theta > 0$.

**(a)** Find the MLE of $\theta$.

**(b)** If $\theta \sim$ Gamma($\alpha, \beta$) so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta)$$

and $E(\theta) = \alpha/\beta$, find the posterior distribution of $\theta$.

**(c)** Find the Bayes estimator of $\theta$ for squared error loss. Is it consistent?

**(d)** What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?

8. (36 points) Suppose that $X_1, \ldots, X_m$ are i.i.d. Geometric($\mu$) and that $Y_1, \ldots, Y_n$ are i.i.d. Geometric($\nu$) and independent of the $X_i$'s. (Thus $P(X_i = x) = \mu(1-\mu)^{x-1}$, $x \in \{1, 2, \ldots\}$ and $P(Y_i = y) = \nu(1-\nu)^{y-1}$ for $y \in \{1, 2, \ldots\}$ where $0 \le \mu, \nu \le 1$. (Reminder: $E(X_1) = 1/\mu$, $Var(X_1) = (1-\mu)/\mu^2$.)

**(a)** Consider testing $H: \ \nu \le \mu$ versus $K: \ \nu > \mu$. Find the joint density function of $(\underline{X}, \underline{Y})$ and show that it can be written in the form of an exponential family with parameter of interest $\theta$ which takes a fixed value $\theta_0$ on the boundary $\Theta_B$ (with an associated statistic $U(\underline{X}, \underline{Y})$), and a nuisance parameter $\xi$ (with an associated sufficient statistic $T(\underline{X}, \underline{Y})$).

**(b)** Find the conditional distribution of $U(\underline{X}, \underline{Y})$ given the sufficient statistic $T(\underline{X}, \underline{Y})$ for $\Theta_B$.

**(c)** Find the UMPU test of $H$ versus $K$, specifying the constants involved as precisely as possible.

**(d)** Describe the Bayes test of $H$ versus $K$ for $0-1$ loss assuming a prior distribution $\Lambda$ of $(\mu, \nu)$. Can you describe the rejection region of this test when $\Lambda$ is the product of two independent Beta priors (i.e. $\mu \sim$ Beta$(\alpha, \beta)$ and $\nu \sim$ Beta$(\gamma, \delta)$)?