Statistics 582, Practice Midterm Exam

Wellner; 2/9/2018

1. (24 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.
   (a) An *admissible* decision rule.
   (b) A *Bayes rule* with respect to a prior distribution Λ.
   (c) A *minimax decision rule*.
   (d) A *least favorable prior distribution*.
   (e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.
   (f) The *Kullback-Leibler information* $K(P, Q)$ between two probability distributions P and Q on a measurable space $(\mathcal{X}, \mathcal{A})$.

2. (24 points) **State** *any two* of the following results:
   (a) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.
   (b) Any theorem / result about nonparametric nonparametric maximum likelhood estimation.
   (d) A uniform strong law of large numbers (or Glivenko - Cantelli theorem).
   (e) Wald's theorem on strong consistency of maximum likelihood estimators.

   Do **either** problem 3 **or** problem 4.

3. (30 points) Suppose that $(X|\theta) \sim \text{Poisson}(\theta)$,

$$p(x|\theta) = e^{-\theta}\frac{\theta^x}{x!}, \qquad x \in \{0, 1, 2, \ldots\},$$

   and the prior distribution of θ is Gamma(α, β), i.e.

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta).$$

   (a) find the posterior distribution of θ.
   (b) Find the Bayes estimator of θ for squared error loss, $L(\theta, a) = (\theta - a)^2$.
   (c) Find the Bayes estimator for testing $H_0 : \ \theta \in (0, 3]$ versus $H_1 : \ \theta \in (3, \infty)$.
   (d) Find the Bayes estimator of θ for the loss function $L(\theta, a) = (\theta - a)^2/\theta$.

4. (30 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. with mixture density (mass function)

$$p(x; \lambda, \mu, \theta) = \theta \frac{\lambda^x}{x!} e^{-\lambda} + (1-\theta) \frac{\mu^x}{x!} e^{-\mu}\,, x = 0, 1, \ldots,$$

where $0 < \theta < 1$, $0 < \lambda \neq \mu < \infty$; in other words, p is the mixture of two Poisson distributions with parameters λ and μ respectively.
A. Describe an EM - algorithm for estimation of (λ, μ, θ).
B. What is the natural corresponding nonparametric model for the data which were modeled with the parametric mixture distribution in A? What is the natural nonparametric maximum likelihood estimator here?

5. (30 points) Suppose that (X_i, Y_i), $i = 1, \ldots, n$ are independent pairs of random variables with

$$X_i \sim \text{exponential}(\beta_i/\alpha)\,, \qquad Y_i \sim \text{exponential}(1/\beta_i \alpha)$$

independent. Here $\alpha > 0$ and $\beta_i > 0$ for $i = 1, \ldots, n$ are all unknown. Thus the joint density of (X_i, Y_i) is

$$f_{X_i,Y_i}(x_i, y_i) = \alpha^{-2} \exp(-\beta_i x/\alpha) \exp(-y_i/\alpha\beta_i) 1_{[0,\infty)}(x_i) 1_{[0,\infty)}(y_i)\,.$$

A. Find the maximum likelihood estimator $\widehat{\alpha}_n$ of α.
B. Do our theorems about consistency and asymptotic normality of maximum likelihood estimators apply to $\widehat{\alpha}_n$? Why or why not? To what (famous) model is the above model analogous?
C. Compute $E\sqrt{X_i}$, $E\sqrt{Y_i}$, and use these together with independence of X_i and Y_i to compute $E\sqrt{X_i Y_i}$. Also compute $Var(\sqrt{X_i Y_i})$. [Hint: $\Gamma(1/2) = \sqrt{\pi}$.]
D. use the results of C to show that $\hat{\alpha}_n \to_p c\alpha$ for some constant c and identify the constant c.