

Statistics 582, Problem Set 8 Solutions

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1. Consider the Locally Most Powerful test ϕ for testing $H : \theta \leq 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.

(a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.

(b) What is the behavior of each of these two approximations for large values of θ ? Which of them shows that the power function decreases to 0 as $\theta \rightarrow \infty$? Why?

Solution: (a) The test is “reject H if $\sqrt{n}\bar{Y}_n > 2^{-1/2}z_\alpha$ ” where $Y_i \equiv 2X_i/(1+X_i^2)$ are i.i.d. and $X_i \sim \text{Cauchy}(\theta, 1)$. Thus under P_θ , by using contour integration and Cauchy’s formula, or by using Mathematica, Maple, or your favorite symbolic manipulation program,

$$\begin{aligned} m(\theta) \equiv E_\theta Y_i &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2} p_\theta(x) dx = \int_{-\infty}^{\infty} \frac{2x}{1+x^2} \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} dx \\ &= \frac{2\theta}{4+\theta^2}, \end{aligned}$$

and

$$\begin{aligned} \sigma^2(\theta) \equiv \text{Var}_\theta(Y_i) &= E_\theta Y_i^2 - m^2(\theta) = \frac{2(4+3\theta^2)}{(4+\theta^2)^2} - \left(\frac{2\theta}{4+\theta^2}\right)^2 \\ &= \frac{2}{4+\theta^2}. \end{aligned}$$

For local alternatives $\theta = \theta_n = t/\sqrt{n}$, we have

$$\begin{aligned} \text{Power}(\theta_n) &= P_{\theta_n}(\sqrt{n}\bar{Y}_n > 2^{-1/2}z_\alpha) \\ &= P_{\theta_n}(\sqrt{n}(\bar{Y}_n - m(\theta_n)) \geq 2^{-1/2}z_\alpha - \sqrt{n}(m(\theta_n) - m(0))) \\ &\rightarrow P(2^{-1/2}Z \geq 2^{-1/2}z_\alpha - m'(0)t) \end{aligned}$$

where

$$\begin{aligned} m'(0) &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2} \frac{d}{d\theta} p_\theta(x) |_{\theta=0} dx \\ &= \int_{-\infty}^{\infty} \dot{l}_\theta(x; 0) \dot{l}_\theta(x; 0) p_\theta(x; 0) dx = I(\theta) = 1/2. \end{aligned}$$

Hence we have

$$\text{Power}(\theta_n) \rightarrow P(Z > z_\alpha - 2^{-1/2}t) = 1 - \Phi(z_\alpha - 2^{-1/2}t).$$

This approximation to the power function increases monotonically from α at $t = 0$ to 1 at $t = \infty$ (effectively when $t > 2^{1/2} \cdot 4$). Note that this result is very much in qualitative agreement with corollary 4.2.4 from Statistics 581.

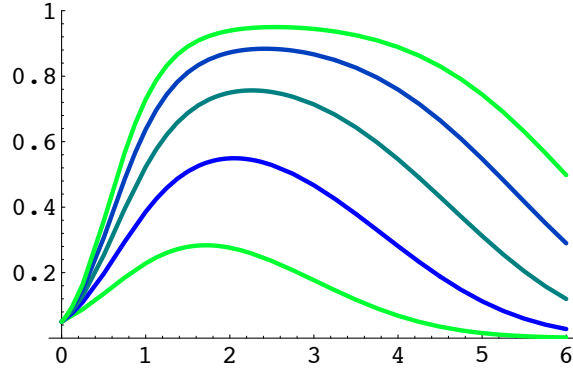


Figure 1: Plots of fixed θ power approximations for $n = 3, 6, 9, 12, 15$.

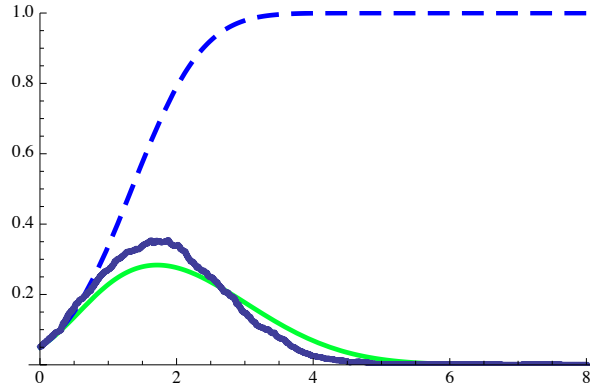


Figure 2: Local (dashed) and fixed (green) θ power approximations, $n = 3$; Monte Carlo estimate of true power (blue)

(b) For fixed alternatives $\theta > 0$ we have

$$\begin{aligned} \text{Power}(\theta) &= P_\theta(\sqrt{n}\bar{Y} > 2^{-1/2}z_\alpha) \\ &= P_\theta(\sqrt{n}(\bar{Y}_n - m(\theta)) > 2^{-1/2}z_\alpha - \sqrt{nm}(\theta)) \\ &\doteq P(Z > (2^{-1/2}z_\alpha - \sqrt{nm}(\theta))/\sigma(\theta)) \\ &= 1 - \Phi((2^{-1/2}z_\alpha - \sqrt{nm}(\theta))/\sigma(\theta)). \end{aligned}$$

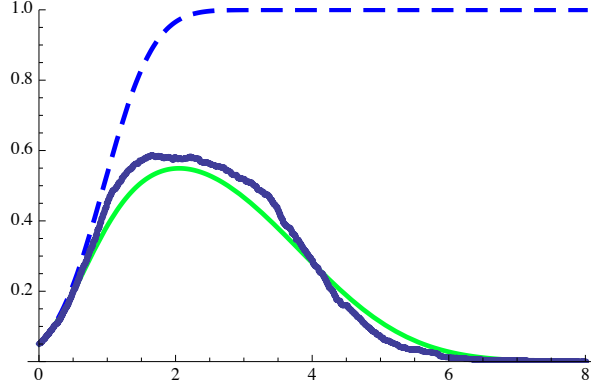


Figure 3: Local (dashed) and fixed (green) θ power approximations, $n = 6$; Monte Carlo estimate of true power (blue)

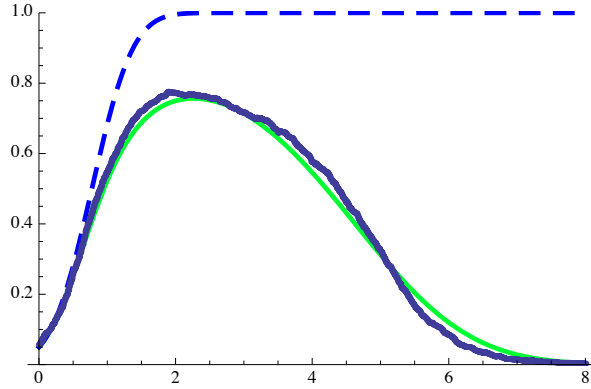


Figure 4: Local (dashed) and fixed (green) θ power approximations, $n = 9$; Monte Carlo estimate of true power (blue)

where

$$\frac{m(\theta)}{\sigma(\theta)} = \frac{\sqrt{2\theta^2}}{\sqrt{4 + \theta^2}}$$

which increases from 0 at $\theta = 0$ to a maximum of $1/\sqrt{2}$ at $\theta = 2$, and then tends to $\sqrt{2}$ as θ increases to ∞ . Thus our fixed alternative approximation to the power function is completely determined by the function $(2^{-1/2}z_\alpha - \sqrt{nm}(\theta))/\sigma(\theta)$, and this first decreases as θ increases (as it should if the power is to increase), but then it reaches a minimum and increases thereafter (approximately as $2^{-1}z_\alpha\theta - \sqrt{2n}$ as $\theta \rightarrow \infty$). Hence this approximation to the power decreases to zero just as we argued that it must in class. See Figures 1-6 which give plots of the two approximations for $\alpha = .05$ and $n = 3, 6, 9, 12, 15$, together with Monte-Carlo estimates of the true power function based on $m = 1000$ Monte-Carlo replica-

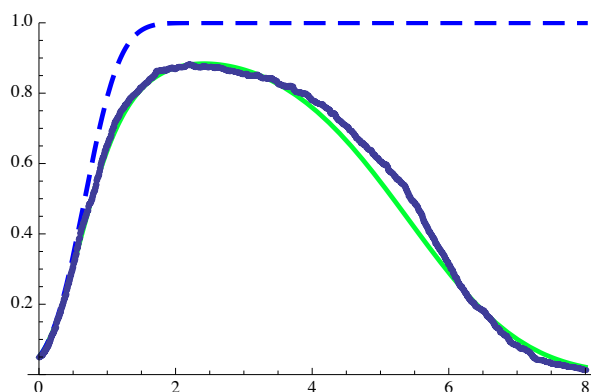


Figure 5: Local (dashed) and fixed (green) θ power approximations, $n = 12$; Monte Carlo estimate of true power (blue)

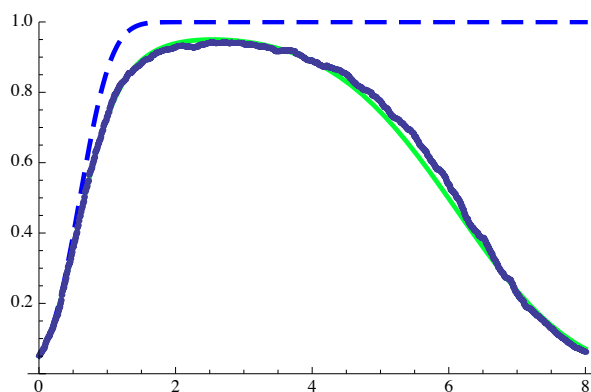


Figure 6: Local (dashed) and fixed (green) θ power approximations, $n = 15$; Monte Carlo estimate of true power (blue)

tions at each sample size (dark blue). Note that the two approximations agree for θ 's close to 0, but the local approximation is always monotone increasing, while the approximations with θ fixed show the approximate power decreasing to 0 as $\theta \rightarrow \infty$ as we know it must. Also note that the fixed alternative approximation to the power function based on the CLT is reasonably accurate for $n \geq 9$.

2. This is a continuation of the previous problem.
 - (a) Suppose you decide to base a test of the hypotheses in the setting of Example 6.1.5 on the sample mean \bar{X}_n . Show how to carry out this test at exact size $0 < \alpha < 1/2$, and compute its power function $\beta_\phi(\theta)$. [Hint: Recall that for $\theta_0 = 0$, and Cauchy data, $\bar{X}_n \stackrel{d}{=} X_1$.]
 - (b) Alternatively, consider a test of the same hypotheses on the sample median

$\mathbb{F}_n^{-1}(1/2)$, or perhaps more simply, the proportion of observations exceeding $\theta_0 = 0$. Show how to carry out this test at exact and approximate size $\alpha \in (0, 1/2)$. Compute the power function of your new test and show that it has a monotone non-decreasing power function $\beta_\phi(\theta)$.

(c) Compare the power functions (or best approximations thereof) for the tests in (a) and (b) and problem 1.

Solution: (a) Consider $\bar{X}_n \equiv n^{-1} \sum_{i=1}^n X_i$. Then, since $X_i \stackrel{d}{=} V_i + \theta$ where V_i are i.i.d. standard Cauchy random variables with $E e^{itV_1} = \exp(-|t|)$ (see e.g. Shorack, *Probability for Statisticians*, page 343), it follows that

$$\begin{aligned} \phi_{\bar{X}_n}(t) &= E \exp(itn^{-1} \sum_1^n X_i) = \phi_{X_1}(t/n)^n = (E \exp(itn^{-1} X_1))^n \\ &= (E \exp(itn^{-1}(V_1 + \theta)))^n = \phi_{V_1}(t/n)^n \cdot e^{it\theta} = \\ &= \exp(-|t|) e^{it\theta} = \phi_{X_1}(t); \end{aligned}$$

i.e. $\bar{X}_n \stackrel{d}{=} X_1$. Thus the test $\phi(\underline{X}) = 1\{\bar{X}_n > k\}$ has

$$\alpha = E_0 \phi(\underline{X}) = P_0(\bar{X}_n > k) = P_0(X_1 > k)$$

if $k = \tan((1/2 - \alpha)\pi) \equiv k_\alpha$, and then

$$\begin{aligned} \beta_\phi(\theta) &= P_\theta(\bar{X}_n > k_\alpha) = P_\theta(X_1 - \theta > k_\alpha - \theta) \\ &= P_0(V_1 > k_\alpha - \theta) = \int_{k_\alpha - \theta}^{\infty} \frac{1}{\pi} \frac{1}{1 + y^2} dy \end{aligned}$$

which is monotone increasing as a function of θ ; see Figure 7. On the other hand, this test does not yield increasing power as the sample size n increases: it has the same power function for all n !

(b) Let $Z_i \equiv 1\{X_i > 0\}$ for $i = 1, \dots, n$. Thus the Z_i 's are independent Bernoulli $p = p_\theta$ random variables with

$$\begin{aligned} p_\theta &= P_\theta(X_i > 0) = P_\theta(X_i - \theta > -\theta) = P_0(Y_i > -\theta) = \int_{-\theta}^{\infty} \frac{1}{\pi} \frac{1}{1 + y^2} dy \\ &\nearrow 1 \text{ as } \theta \rightarrow \infty. \end{aligned}$$

Note that testing $H : \theta \leq \theta_0 = 0$ versus $K : \theta > 0$ is equivalent to testing $H' : p \leq p_0 \equiv 1/2$ versus $K' : p > p_0 = 1/2$. Now

$$T_n \equiv \sum_1^n Z_i = n(1 - \mathbb{F}_n(0)) \sim \text{Binomial}(n, p)$$

is a one-parameter exponential family with density function $p(\cdot; p)$ with respect to counting measure on \mathbb{N} given by

$$\begin{aligned} p(y; p) &= P_p(T_n = y) = \binom{n}{y} p^y (1-p)^{n-y} = \binom{n}{y} \left(\frac{p}{1-p} \right)^y (1-p)^n \\ &= c(p) \exp(Q(p)y) h(y) \end{aligned}$$

with $Q(p) \equiv \log(p/(1-p))$, $c(p) \equiv (1-p)^n$, and $h(y) = \binom{n}{y}$, so it has monotone likelihood ratio in y . Thus by the Karlin-Rubin theorem, the test

$$\varphi(\underline{X}) = \phi(\underline{Z}) = \begin{cases} 1 & \text{if } T_n = \sum_1^n Z_i > k \\ \gamma & \text{if } T_n = k \\ 0 & \text{if } T_n < k, \end{cases}$$

where k and γ are chosen so that $E_{p_0=1/2} \phi(\underline{Z}) = \alpha$, has power function

$$\beta_\phi(p_\theta) \equiv \tilde{\beta}_\varphi(\theta)$$

which is monotone increasing in p_θ and θ . This test improves with increasing sample size n ; see Figure 7

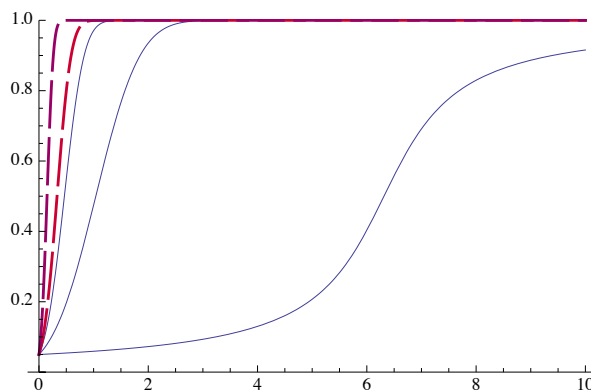


Figure 7: Plot comparing local power approximations for the Binomial test (solid), locally most powerful test (dashed), and the mean test, for $n = 10, 50$.

3. Let X and Y be random variables with joint density

$$p_{X,Y}(x, y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y).$$

- Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \leq \mu + 1$ versus $H_1 : \lambda > \mu + 1$.
- Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda = \mu$ versus $H_1 : \lambda \neq \mu$.

(c) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \geq 2\mu$ versus $H_1 : \lambda < 2\mu$.

(d) What happens when X_1, \dots, X_m are i.i.d. $\text{Exponential}(\lambda)$ and Y_1, \dots, Y_n are i.i.d. $\text{Exponential}(\mu)$?

Solution: When $X \sim \text{exp}(\lambda)$ and $Y \sim \text{exp}(\mu)$ we have

$$p_{\lambda,\mu}(x, y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y).$$

(a) For testing $H : \lambda \leq \mu + 1$ versus $K : \lambda > \mu + 1$ we rewrite the density as follows:

$$\begin{aligned} p_{\lambda,\mu}(x, y) &= \lambda\mu \exp(-\lambda x - \mu y) \\ &= \lambda\mu \exp((\lambda - \mu)y - \lambda(x + y)) \\ &= \lambda\mu \exp(\theta U(x, y) + \xi T(x, y)) \end{aligned}$$

where $\theta \equiv \lambda - \mu$, $U(x, y) \equiv y$, $\xi \equiv -\lambda$, and $T(x, y) \equiv x + y$. Since $\lambda \leq \mu + 1$ is equivalent to $\lambda - \mu = \theta \leq 1 \equiv \theta_0$, our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } Y > c_\alpha(T) \\ \gamma(T) & \text{if } Y = c_\alpha(T) \\ 0 & \text{if } Y < c_\alpha(T) \end{cases}$$

where c_α and $\gamma(\alpha)$ satisfy $E\{\phi(X, Y)|T\} = \alpha$. In this case, the conditional distribution of Y given $T = X + Y$ on the boundary $\Theta_B = \{(\lambda - 1, \lambda) : \lambda \geq 1\}$ is given by

$$f_{Y|T}(y|t) = \frac{e^y}{e^t - 1} 1_{[0,t]}(y).$$

Therefore $1 - F_{Y|T}(y|t) = 1 - (e^y - 1)/(e^t - 1)$ and for $\alpha = .2$ the critical point for the conditional test is given by

$$c_\alpha(T) = \log(\exp(T) - (\exp(T) - 1)/5) = \log((4/5)\exp(T) + 1/5), \quad \gamma(T) = 0.$$

(b) For testing $H : \lambda = \mu$ versus $K : \lambda \neq \mu$, the same rewrite of the density as in (a) works. Now we have $\lambda = \mu$ is equivalent to $\mu - \lambda = 0 \equiv \theta_0$, and $\lambda \neq \mu$ is equivalent to $\mu - \lambda \neq 0 \equiv \theta_0$, so our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } Y > c_2(T) \text{ or } Y < c_1(T) \\ \gamma_i(T) & \text{if } Y = c_1(T) \text{ or } Y = c_2(T) \\ 0 & \text{if } c_1(T) < Y < c_2(T) \end{cases}$$

where the c_1 , c_2 , γ_1 and γ_2 are determined so that $E\{\phi(X, Y)|T\} = \alpha$. In this case the conditional distribution of Y given T on $\Theta_B = \{(\lambda, \lambda) : \lambda \geq 0\}$ is Uniform(0, T), and hence the conditional distribution of Y/T given T is Uniform(0, 1), and this is independent of T . Hence the UMPU test of H versus K of size .2 is given by “reject H if $Y/T < .1$ or $Y/T > .9$ ”.

(c) For testing $H : \lambda \geq 2\mu$ versus $K : \lambda < 2\mu$, we need a somewhat different rewrite of the joint density. Now

$$\begin{aligned} p_{\lambda, \mu}(x, y) &= \lambda\mu \exp(-\lambda x - \mu y) \\ &= \lambda\mu \exp(-(\lambda - 2\mu)x - \mu(2x + y)) \\ &= \lambda\mu \exp(\theta U(x, y) + \xi T(x, y)) \end{aligned}$$

where $\theta \equiv 2\mu - \lambda$, $U(x, y) \equiv x$, $\xi \equiv -\mu$, and $T(x, y) \equiv 2x + y$. Since $\lambda \geq 2\mu$ is equivalent to $2\mu - \lambda \equiv \theta \leq 0 \equiv \theta_0$, (and $\lambda < 2\mu$ is equivalent to $2\mu - \lambda = \theta > 0 \equiv \theta_0$), our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } X > c_\alpha(T) \\ \gamma(T) & \text{if } X = c_\alpha(T) \\ 0 & \text{if } X < c_\alpha(T) \end{cases}$$

where $c(T)$ and $\gamma(T)$ satisfy $E\{\phi(X, Y)|T\} = \alpha$. In this case the conditional distribution of X given T is Uniform(0, $T/2$), so $2X/T$ is Uniform(0, 1) and independent of T . Hence the UMPU test of H versus K of size $\alpha = .2$ is given by “reject H if $2X/T > .8$ ”.

When we observe X_1, \dots, X_m are i.i.d. Exponential(λ) and Y_1, \dots, Y_n are i.i.d. Exponential(μ), then the distribution of the observations is given by

$$\begin{aligned} p_{\lambda, \mu}(\underline{x}, \underline{y}) &= \lambda^m \mu^n \exp(-\lambda \sum_{i=1}^m x_i - \mu \sum_{j=1}^n y_j) \\ &= \lambda^m \mu^n \exp((\lambda - \mu) \sum_{j=1}^n y_j - \lambda(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j)) \\ &= \lambda^m \mu^n \exp(\theta U(\underline{x}, \underline{y}) + \xi T(\underline{x}, \underline{y})) \end{aligned}$$

where $\theta \equiv \lambda - \mu$, $U(\underline{x}, \underline{y}) \equiv \sum y_j$, $\xi \equiv -\lambda$, and $T(\underline{x}, \underline{y}) \equiv \sum x_i + \sum y_j$. This rewrite works for (a) and (b), and a similar rewrite works for (c) with the new $U = \sum x_i$, $T = 2\sum x_i + \sum y_j$. The form of the tests in (a) - (c) remains the same with the new U and T , and all that remains is to calculate the conditional distributions of U given T . In (a) this density is given by

$$f_{U|T}(u|t) = \frac{u^{n-1}(t-u)^{m-1}e^{-u}}{\int_0^t v^{n-1}(t-v)^{m-1}e^{-v} dv}.$$

In (b) it is easily found that $V \equiv U/T \sim \text{Beta}(n, m)$, and the test can be carried out unconditionally using tables of the Beta distributions. In (c) $2U \equiv \sum 2X_i \sim \text{Gamma}(m, \mu)$ and $V \equiv \sum Y_j \sim \text{Gamma}(n, \mu)$ on the boundary $\lambda = 2\mu$, so $2\mu U \sim \chi_{2m}^2$ and $\mu V \sim \chi_{2n}^2$. Therefore

$$\frac{2U/(2m)}{V/(2n)} = \frac{n}{m} 2 \frac{U}{V} \sim F_{2m, 2n}$$

and since $2U/T = 2U/(2U + V) = (2U/V)(1 + (2U/V))$ is a monotone increasing function of $2U/V$, the UMPU test can be carried out unconditionally using tables of the $F_{2m, 2n}$ distributions.

4. Lehmann and Romano, TSH, problem 4.3, page 139, modified: Let $X \sim \text{Binomial}(n, p)$, and consider testing $H : p = p_0$ versus $K : p \neq p_0$ at level $\alpha = \alpha$. Determine the boundary values of the UMP unbiased test for $n = 20$ with $\alpha = .05$, $p_0 = .2$, and with $\alpha = .10$, $p_0 = .4$. In each case plot the power functions of both the unbiased and the equal-tails test.

Solution: For the first scenario: $n = 20$, $\alpha = .05$, $p_0 = .2$, we find, using the calculations in Lehmann and Romano, TSH, page 113, the UMP unbiased test is $\phi(x) = \gamma_1 1\{x = 1\} + \gamma_2 1\{x = 8\} + 1\{x \geq 9\} + 1\{x=0\}$ with $c_1 = 1$, $c_2 = 8$, $\gamma_1 = .02999$, and $\gamma_2 = .5054$. The corresponding equal tails test has $c_1 = 1$, $c_2 = 8$, $\gamma_1 = 0.2337$, and $\gamma_2 = 0.6777$. The power functions of these two tests are shown in Figures 8 and 9.

For the second scenario: $n = 20$, $\alpha = .10$, $p_0 = .4$, we find, using the calculations in Lehmann and Romano, TSH, page 113, the UMP unbiased test is $\phi(x) = \gamma_1 1\{x = 5\} + \gamma_2 1\{x = 12\} + 1\{x \geq 13\} + 1\{x \leq 4\}$ with $c_1 = 5$, $c_2 = 12$, $\gamma_1 = 0.0033$, and $\gamma_2 = 0.782377$.

The corresponding equal tails test has $c_1 = 4$, $c_2 = 12$, $\gamma_1 = 0.9738$, and $\gamma_2 = 0.8161$. The power functions of these two tests are shown in Figure 10; since the binomial distribution is quite close to being symmetric for $p = .4$, they do not differ substantially.

5. **Optional bonus problem 1:** Let $\mathcal{P} = \{p_\theta : \theta \in \Theta\}$ where p_θ is a family of densities with respect to a fixed dominating measure μ defined on a sample space \mathcal{X} .

(a) Suppose that the densities $p_\theta(x) \equiv p(x, \theta)$ have a second mixed partial derivative and that

$$\frac{\partial}{\partial x} \frac{\partial}{\partial \theta} \log p(x, \theta) \geq 0$$

for all $x \in \mathbb{R}$ and $\theta \in \Theta$. Show that the inequality in the last display implies that \mathcal{P} has monotone likelihood ratio. [Hint: use the fundamental theorem of calculus

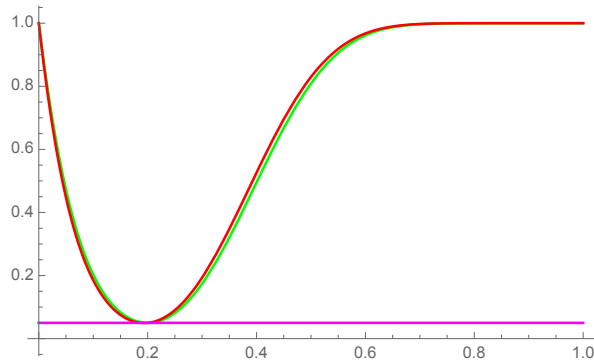


Figure 8: Power functions, UMP unbiased test (green) and equal tails test (red): $n = 20$, $p_0 = .2$, $\alpha = .05$

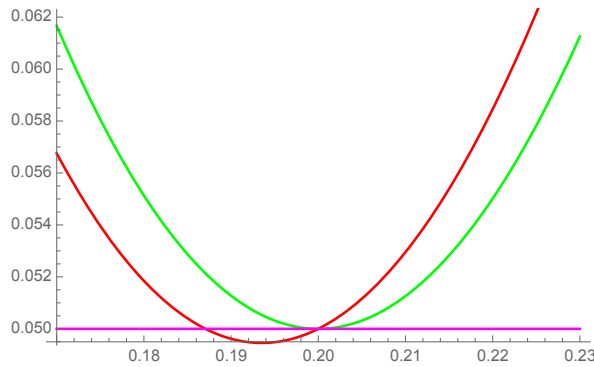


Figure 9: Power functions, UMP unbiased test (green) and equal tails test (red): $n = 20$, $p_0 = .2$, $\alpha = .05$

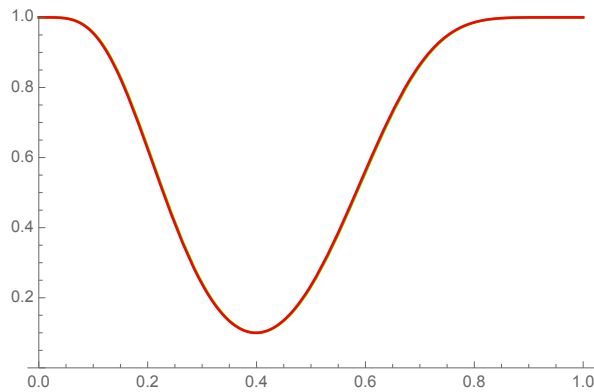


Figure 10: Power functions, UMP unbiased test (green) and equal tails test (red): $n = 20$, $p_0 = .4$, $\alpha = .10$

twice.] (b) Show that the condition in (a) is equivalent to

$$p(x, \theta) \frac{\partial}{\partial x} \frac{\partial}{\partial \theta} p(x, \theta) \geq \frac{\partial}{\partial \theta} p(x, \theta) \frac{\partial}{\partial x} p(x, \theta) \quad \text{for all } \theta, x.$$

Solution: (a) \mathcal{P} has monotone likelihood ratio in x if for each $\theta_1 < \theta_2$, $\theta_1, \theta_2 \in \Theta$, the ratio $p_{\theta_2}(x)/p_{\theta_1}(x)$ is an increasing function of x .

(b) Let $\theta_1 < \theta_2$ and $x_1 < x_2$. Then integrating with respect to x across the given inequality (from x_1 to x_2) yields

$$\begin{aligned} 0 &\leq \int_{x_1}^{x_2} \frac{\partial}{\partial x} \frac{\partial}{\partial \theta} \log p(x, \theta) dx \\ &= \frac{\partial}{\partial \theta} \log p(x_2, \theta) - \frac{\partial}{\partial \theta} \log p(x_1, \theta) \\ &= \frac{\partial}{\partial \theta} \log \frac{p(x_2, \theta)}{p(x_1, \theta)} \quad \text{for all } \theta. \end{aligned}$$

Integrating this last inequality with respect to θ (from θ_1 to θ_2) yields

$$\begin{aligned} 0 &\leq \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \log \frac{p(x_2, \theta)}{p(x_1, \theta)} d\theta \\ &= \log \frac{p(x_2, \theta_2)}{p(x_1, \theta_2)} - \log \frac{p(x_2, \theta_1)}{p(x_1, \theta_1)} \\ &= \log \frac{p(x_2, \theta_2)/p(x_2, \theta_1)}{p(x_1, \theta_2)/p(x_1, \theta_1)}. \end{aligned}$$

Thus it follows by exponentiating both sides of the last display that

$$1 \leq \frac{p(x_2, \theta_2)/p(x_2, \theta_1)}{p(x_1, \theta_2)/p(x_1, \theta_1)},$$

and hence that $p(x_1, \theta_2)/p(x_1, \theta_1) \leq p(x_2, \theta_2)/p(x_2, \theta_1)$. We conclude that \mathcal{P} has MLR if the stated condition on the second mixed partial derivative holds.

(b) Note that

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial x} \frac{\partial}{\partial \theta} \log p(x, \theta) = \frac{\partial}{\partial x} \frac{\frac{\partial}{\partial \theta} p(x, \theta)}{p(x, \theta)} \\ &= \frac{\frac{\partial^2}{\partial x \partial \theta} p(x, \theta)}{p(x, \theta)} - \frac{\frac{\partial}{\partial \theta} p(x, \theta) \frac{\partial}{\partial x} p(x, \theta)}{p(x, \theta)^2}, \end{aligned}$$

so multiplication by $p(x, \theta)^2$ and rearranging yields the claimed equivalence.

6. **Optional bonus problem 2:** Let X_1, \dots, X_n be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.

(a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

(b) Find the subclass of the tests that are independent of θ_1 . These are UMP tests of H_0 versus $H'_1 : \theta > \theta_0$.

(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is UMP of size α for testing $H'_0 : \theta \leq \theta_0$ versus $H'_1 : \theta > \theta_0$ but that ϕ is not admissible.

(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \leq b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \neq \theta_0$.

Solution: (a) For testing $\theta = \theta_0$ versus $\theta = \theta_1 > \theta_0$, the class of all NP tests is given by tests of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) > k \theta_0^{-n} 1_{[0, \theta_0]}(t) \\ \gamma(t), & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) = k \theta_0^{-n} 1_{[0, \theta_0]}(t) \\ 0, & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) < k \theta_0^{-n} 1_{[0, \theta_0]}(t) \end{cases} .$$

Thus for $k = (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \leq \theta_1 \\ \gamma(t), & \text{if else;} \end{cases}$$

for $k > (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \leq \theta_1 \\ \gamma(t), & \text{if else} \\ 0, & \text{if } 0 \leq t \leq \theta_0; \end{cases}$$

and for $k < (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \theta_1 \\ \gamma(t), & \text{if else.} \end{cases}$$

(b) The subclass of these tests that do not depend on θ_1 is the class of tests ϕ with

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t < \infty \\ \gamma(t), & \text{if else} \end{cases}$$

where $E_{\theta_0} \gamma(T) = \alpha$.

(c) The test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is of the form of the tests in (b) with $\gamma(t) = \alpha$ for $0 \leq t \leq \theta_0$, and hence is UMP of size α for testing $\theta = \theta_0$ versus $\theta > \theta_0$. To see that it is UMP for testing $\theta \leq \theta_0$ versus $\theta > \theta_0$, we first compute

its power function to confirm that it is of size α for the composite null hypothesis $\theta \leq \theta_0$. The power is

$$\begin{aligned}\beta_\phi(\theta) &= E_\theta\phi(T) = P_\theta(T > \theta_0) + \alpha P_\theta(T \leq \theta_0) \\ &= \left\{1 - \left(\frac{\theta_0}{\theta}\right)^n\right\}1_{(\theta_0, \infty)}(\theta) + \alpha 1_{[0, \theta_0]}(\theta) + \alpha \left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0, \infty)}(\theta) \\ &= \left\{1 - \left(\frac{\theta_0}{\theta}\right)^n(1 - \alpha)\right\}1_{(\theta_0, \infty)}(\theta) + \alpha 1_{[0, \theta_0]}(\theta).\end{aligned}$$

Thus we see that $\beta_\phi(\theta) = \alpha$ for $\theta \leq \theta_0$, and ϕ is of size α for $\theta \leq \theta_0$. Since the class of size α tests for testing $\theta = \theta_0$ is a larger class than the class of size α tests for testing $\theta \leq \theta_0$, and since ϕ is UMP in the larger class, it is also UMP in the smaller class. Hence ϕ is UMP for testing $\theta \leq \theta_0$ versus $\theta > \theta_0$. But the competing test $\phi_0(t) = 1\{t > (1 - \alpha)^{1/n}\theta_0\}$ has power function

$$\begin{aligned}\beta_{\phi_0}(\theta) &= E_\theta\phi_0(T) = P_\theta(T > \theta_0(1 - \alpha)^{1/n}) \\ &= 1 - \left\{(1 - \alpha) \left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0(1 - \alpha)^{1/n}, \infty)}(\theta) + 1_{[0, \theta_0(1 - \alpha)^{1/n}]}(\theta)\right\},\end{aligned}$$

so the power function of the test ϕ_0 is strictly below that of the test ϕ on the set $[0, \theta_0)$. Hence ϕ is inadmissible and the test ϕ_0 is also UMP.

(d) The test $\phi(t) = 1 - 1_{(\theta_0\alpha^{1/n}, \theta_0]}(t)$ is of size α for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$ since

$$E_{\theta_0}\phi(T) = P_{\theta_0}(T \leq \theta_0\alpha^{1/n}) = \left(\frac{\theta_0\alpha^{1/n}}{\theta_0}\right)^n = \alpha.$$

Furthermore it is of the form of the class of all UMP tests for testing $\theta = \theta_0$ versus $\theta > \theta_0$, hence it UMP among the size α tests for $\theta > \theta_0$. For testing $\theta = \theta_0$ versus $\theta = \theta_1 < \theta_0$, the NP Pearson tests of the form $\phi_1(t) = \gamma(t)1_{[0, \theta_0]}(t)$ are most powerful of their size. But the test ϕ is of this form (with $\gamma(t) = 1_{[0, \theta_0\alpha^{1/n}]}(t)$), is in this class and does not depend on $\theta_1 < \theta_0$. Hence ϕ is UMP for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$. [This is an unusual situation in which we get “something for free” from the structure of the uniform distributions. Usually two-sided tests are *not* UMP!]

7. Optional bonus problem 3: let X and Y be independent random variables with geometric distributions

$$p_{X,Y}(x, y | \theta_1, \theta_2) = (1 - \theta_1)(1 - \theta_2)\theta_1^x\theta_2^y, \quad x, y \in \{0, 1, \dots\}.$$

where $0 < \theta_j < 1$, $j = 1, 2$. Find a UMP unbiased test of size $\alpha = .20$ for testing

(a) $H_0 : \theta_1 \leq \theta_2$ versus $H_1 : \theta_1 > \theta_2$.

(b) $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

(c) For what functions $\varphi(\theta_1, \theta_2)$ do our methods guarantee existence of a UMP unbiased test of $H_0 : \varphi(\theta_1, \theta_2) = 0$ versus $H_1 : \varphi(\theta_1, \theta_2) \neq 0$?

Solution: When X and Y are independent geometric random variables, the joint density (with respect to the product of counting measure on the non-negative integers) can be rewritten as

$$\begin{aligned} P_{\theta_1, \theta_2}(X = x, Y = y) &= (1 - \theta_1)(1 - \theta_2) \exp(x \log \theta_1 + y \log \theta_2) \\ &= (1 - \theta_1)(1 - \theta_2) \exp(x[\log \theta_1 - \log \theta_2] + (x + y) \log \theta_2) \\ &= (1 - \theta_1)(1 - \theta_2) \exp(\theta U + T\xi) \end{aligned}$$

where $U(x, y) = x$, $\theta = \log(\theta_1/\theta_2)$, $T(x, y) = x + y$, and $\xi = \log \theta_2$.

(a) Testing $H : \theta_1 \leq \theta_2$ versus $K : \theta_1 > \theta_2$ in this family is equivalent to testing

$$H : \theta \leq 0, \xi = \text{anything} \quad \text{versus} \quad K : \theta > 0, \xi = \text{anything}.$$

On the boundary $\Theta_B = \{(\theta_1, \theta_2) : \theta_1 = \theta_2\}$, $T = X + Y$ is sufficient and complete. Hence by Theorem 2.4.2 the UMPU test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } X > c(T) \\ \gamma(T) & \text{if } X = c(T) \\ 0 & \text{if } X < c(T) \end{cases}$$

where c and γ are determined by

$$E_{\theta_0=0}\{\phi(X, Y)|T = t\} = \alpha = .2. \tag{1}$$

Now by direct calculation

$$\begin{aligned} P(X = x|T = t) &= \frac{P_{\theta_1, \theta_2}(X = x, Y = t - x)}{P_{\theta_1, \theta_2}(T = t)} \\ &= \frac{\theta_1^x \theta_2^{t-x}}{\sum_{x'=0}^t \theta_1^{x'} \theta_2^{t-x'}} \\ &= \frac{(\theta_1/\theta_2)^x}{\sum_{x'=0}^t (\theta_1/\theta_2)^{x'}} \\ &= \frac{1}{t+1}, \quad x \in \{0, \dots, t\} \quad \text{when } \theta_1 = \theta_2. \end{aligned}$$

Hence (1) becomes:

$$\frac{t - c(t)}{t + 1} + \frac{\gamma(t)}{t + 1} = .2.$$

Thus we choose

$$c(t) = \inf\{k : t - k \leq .2(t + 1)\}, \quad \gamma(t) = .2(t + 1) - (t - c(t)).$$

(b) Testing $H\theta_1 = \theta_2$ versus $K : \theta_1 \neq \theta_2$ in this family is equivalent to testing

$$H : \theta = 0, \xi = \text{anything}, \quad \text{versus} \quad K : \theta \neq 0, \xi = \text{anything}.$$

Hence by Theorem 2.4.2 the UMPU test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } X > c_2(T) \text{ or } X < c_1(T) \\ \gamma_i(T) & \text{if } X = c_i(T), i = 1, 2 \\ 0 & \text{if } c_1(T) < X < c_2(T). \end{cases}$$

where c_i and γ_i are determined by

$$E_{\theta_0=0}\{\phi(X, Y)|T = t\} = \alpha = .2 \tag{2}$$

and

$$E_{\theta_0=0}\{X\phi(X, Y)|T = t\} = \alpha E_{\theta_0=0}(X|T = t). \tag{3}$$

Since $(X|T = t) \sim$ Discrete Uniform on $\{0, \dots, t\}$, the two equalities in (2) and (3) become

$$c_1(t) + t - c_2(t) + \gamma_1(t) + \gamma_2(t) = .2(t + 1),$$

and, abbreviating $c_i(t) = c_i$, $\gamma_i(t) = \gamma_i$,

$$t(t + 1) + 2c_1\gamma_1 + 2c_2\gamma_2 = \alpha t(t + 1).$$

(c) For functions of the form $g(\theta_1, \theta_2) = a \log \theta_1 + b \log \theta_2 - c$ for fixed numbers a, b, c with at least one of a, b different from 0 we can derive UMPU tests of $H : g(\theta_1, \theta_2) = 0$ versus $K : g(\theta_1, \theta_2) \neq 0$. This can be seen as follows: $g(\theta_1, \theta_2) = 0$ is equivalent to $\log(\theta_1 \theta_2^{b/a}) = c/a$ if $a \neq 0$. Then note that the exponential term of the joint density of X, Y can be written as

$$\begin{aligned} \exp(x \log \theta_1 + y \log \theta_2) &= \exp(x \log(\theta_1 \theta_2^{b/a}) + (y - (b/a)x) \log \theta_2) \\ &= \exp(\theta U + \xi T) \end{aligned}$$

with $U(x, y) = x$, $\theta = \log(\theta_1 \theta_2^{b/a})$, $T(x, y) = x + y$, and $\xi = \log \theta_2$.