

Statistics 582, Problem Set 7 Solutions

Wellner; 2/25/2015

1. A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$.54	.08	.12	.26
$p_1(x)$.22	.16	.36	.26

(a) Find a most powerful test of size $\alpha = .15$ for testing p_0 versus p_1 and determine its power.

(b) Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$.

Solution: (a) Now $p_1(x)/p_0(x) = \{22/54 = 11/27, 2, 3, 1\}$, according as $x = 1, 2, 3, 4$, so a MP test of size $\alpha = .15$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 3 \\ .03/.08, & \text{if } x = 2 \\ 0, & \text{if } x = 1, 4. \end{cases}$$

Then

$$E_0\phi(X) = P_0(X = 3) + \frac{.03}{.08}P_0(X = 2) = .12 + \frac{.03}{.08}.08 = .15,$$

while

$$\text{Power} = E_1\phi(X) = P_1(X = 3) + \frac{.03}{.08}P_1(X = 2) = .36 + \frac{.03}{.08}.16 = .36 + .06 = .42.$$

(b) A Bayes rule (test) ϕ_λ with respect to the prior $\underline{\lambda} = (1/2, 1/2)$ is given by

$$\begin{aligned} \phi_\lambda(x) &= \begin{cases} 1, & \text{if } p_1(x) \geq p_0(x) \\ 0, & \text{if } p_1(x) < p_0(x) \end{cases} \\ &= \begin{cases} 1, & \text{if } x = 2, 3, 4 \\ 0, & \text{if } x = 1. \end{cases} \end{aligned}$$

Then $R(0, \phi_\lambda) = E_0\phi_\lambda(X) = .46$, $R(1, \phi_\lambda) = E_1(1 - \phi_\lambda(X)) = .22$, and hence the Bayes risk with respect to the prior $\lambda = (1/2, 1/2)$ is $\mathcal{R}(\Lambda, \phi_\lambda) = (.46 + .22)/2 = .34$. Note that the sum of the two types of error is

$$R(0, \phi_\lambda) + R(1, \phi_\lambda) = .68 = \int p_0 \wedge p_1 d\mu = 1 - d_{TV}(P_0, P_1).$$

Note that for the Neyman Pearson test with $\alpha = .15$, the sum of the two types of error is $.15 + .58 = .73 > .68$.

2. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0 , P_1 , and P_2 be the probability distributions assigning to the integers $1, \dots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$.03	.02	.02	.01	0	.92
$p_1(x)$.06	.05	.08	.02	.01	.78
$p_2(x)$.09	.05	.12	0	.02	.72

Determine whether there exists a level- α test of $H : P = P_0$ which is UMP against the alternatives P_1 and P_2 when:

- (i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

Solution: Here the table of likelihood ratios is as follows:

x	1	2	3	4	5	6
$p_1(x)/p_0(x)$	2	5/2	4	2	∞	78/98
$p_2(x)/p_0(x)$	3	5/2	6	0	∞	72/98

- (i) For $\alpha = .01$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\begin{aligned}\phi_1(x) &= 1\{x = 5\} + (1/2)1\{x = 3\}, \\ \phi_2(x) &= 1\{x = 5\} + (1/2)1\{x = 3\},\end{aligned}$$

so $\phi_1 = \phi_2$ is Uniformly most powerful.

- (ii) For $\alpha = .05$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\begin{aligned}\phi_1(x) &= 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x), \\ \phi_2(x) &= 1_{\{1,3,5\}}(x),\end{aligned}$$

so there is no UMP test of P_0 versus P_1 and P_2 at this level.

- (iii) For $\alpha = .07$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\begin{aligned}\phi_1(x) &= 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x), \\ \phi_2(x) &= 1_{\{1,2,3,5\}},\end{aligned}$$

so by taking $\gamma(x) = 1\{x = 1\}$, $\phi_1(x) = \phi_2(x)$, and this test is Uniformly Most Powerful for testing P_0 versus P_1 and P_2 .

3. (Problem 3.7, Lehmann and Romano, TSH, page 94, modified) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	$\theta/2$	θ	$.9 - \theta$	$.1 - \theta/2$

where $0 < \theta < .2$. For testing $H : \theta = .10$ against $\theta > .10$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:

- (i) $\phi(0) = 1, \phi(1) = \phi(2) = \phi(3) = 0$;
- (ii) $\phi(1) = .5, \phi(0) = \phi(2) = \phi(3) = 0$;
- (iii) $\phi(3) = 1, \phi(0) = \phi(1) = \phi(2) = 0$.

Solution: The likelihood ratios $P_{\theta'}(X = x)/P_\theta(X = x)$

x	0	1	2	3
$P_\theta(X = x)$	$\theta/2$	θ	$.9 - \theta$	$.1 - \theta/2$
$\frac{P_{\theta'}(X=x)}{P_\theta(X=x)}$	$\frac{\theta'}{\theta}$	$\frac{\theta'}{\theta}$	$\frac{9-10\theta'}{9-10\theta}$	$\frac{1-5\theta'}{1-5\theta}$

It is easy to check that

$$\frac{\theta'}{\theta} > \frac{9 - 10\theta'}{9 - 10\theta} > \frac{1 - 5\theta'}{1 - 5\theta}$$

Hence this family has monotone decreasing likelihood ratio in x (though not strictly), and strictly decreasing likelihood ratio in

$$\begin{aligned} T(x) &= 1\{x = 0\} + 1\{x = 1\} + 2 \cdot 1\{x = 2\} + 3 \cdot 1\{x = 3\} \\ &= x1\{x > 0\} + 1\{x = 0\}. \end{aligned}$$

It follows from the Karlin - Rubin theorem that a UMP test of $H : \theta \leq \theta_0 = .05$ (of its level) is given by

$$\phi(X) = 1_{[T(X) < k]} + \gamma(X)1_{[T(X) = k]}. \quad (1)$$

(i) Note that the test $\phi_1(X) = 1\{X = 0\}$ is of the form (1) with $k = 1$ and $\gamma(X) = 1\{X = 0\}$ and it has level $\alpha = .05$; hence it is a UMP test of H versus K . The power of ϕ_1 is given by $\beta_1(\theta) \equiv E_\theta\phi_1(X) = \theta/2$.

(ii) The test $\phi_2(X) = (.5)1\{X = 1\}$ is also of the form (1) with $k = 1$ and $\gamma(X) = .5 \cdot 1\{X = 1\}$ and it has level $\alpha = .05$. Hence it is also a UMP test of H versus K . The power of ϕ_2 is given by $\beta_2(\theta) \equiv E_\theta\phi_2(X) = \theta/2$.

(iii) The test $\phi_3(X) = 1\{X = 3\}$ is clearly not of the form (1). It has power function $\beta_3(\theta) = E_\theta\phi_3(X) = .1 - \theta/2$, so $\beta_3(.05) = .05$, but $\beta_3(\theta) > .05$ for $\theta < .10$ while $\beta_3(\theta) < .05$ for $\theta > \theta_0 = .10$. In fact, this is a UMP test of $\tilde{H} : \theta \geq \theta_0$ versus $\tilde{K} : \theta < \theta_0$.

4. Suppose that X_1, \dots, X_n are i.i.d. $N(\theta, \sigma^2)$.

(a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ . Find the entire risk body and plot it.

(b) What happens to the risk body as n grows or as $\sigma_0 \rightarrow 0$?

(c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?

(d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : n \geq 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?

Solution: (a) If X_1, \dots, X_n are i.i.d. $N(\theta, \sigma_0)$, to find optimal tests ϕ we can reduce (by sufficiency) to consideration of $\bar{X} \sim N(\theta, \sigma_0^2/n)$. My favorite family of tests (in fact the most powerful tests) of H versus K are the tests $\phi_c(\underline{X}) = 1\{\bar{X} > c\}$. For these tests

$$\begin{aligned} R(0, \phi_c) &= E_0 \phi_c(\underline{X}) = P_0(\bar{X} > c) \\ &= P_0(\sqrt{n}(\bar{X} - 0)/\sigma_0 > \sqrt{nc}/\sigma_0) \\ &= 1 - \Phi(\sqrt{nc}/\sigma_0) \end{aligned}$$

and

$$\begin{aligned} R(1, \phi_c) &= E_1(1 - \phi_c(\underline{X})) \\ &= P_1(\bar{X} \leq c) = P_1(\sqrt{n}(\bar{X} - 1) \leq \sqrt{n}(c - 1)) \\ &= \Phi(\sqrt{n}(c - 1)/\sigma_0). \end{aligned}$$

Since these tests are MP for testing H versus K , there are no points with risks below the curve given by $\{(R(0, \phi_c), R(1, \phi_c)) : c \in \mathbb{R}\}$; this is the lower boundary of the risk body. Note that the tests $\phi_{\text{ignore}}(\underline{X}) \equiv \alpha$ have risks $R(0, \phi_{\text{ignore}}) = \alpha$, $R(1, \phi_{\text{ignore}}) = 1 - \alpha$. Thus the line $\{(\alpha, 1 - \alpha) : \alpha \in [0, 1]\}$ is in the risk body. Furthermore, note that the tests $\phi'_c(\underline{X}) \equiv 1 - \phi_c(\underline{X}) = 1\{\bar{X} \leq c\}$ are MP for testing $H : \theta = 0$ versus $K' : \theta = \theta_1 < 0$, and by the Karlin - Rubin theorem these tests minimize the power function at points $\theta = \theta_1$ in the class of all tests with fixed power function (say at α) at $\theta = \theta_0$. Since

$$\text{Power}_{\phi'_c}(\theta) = E_\theta \phi'_c = 1 - R(\theta, \phi_c),$$

this says that the tests ϕ'_c maximize $R(1, \phi_c)$ over tests ϕ with $R(0, \phi) = \alpha$. Hence there are no points in the risk body with risks above the curve given by $\{(1 - R(0, \phi_c), 1 - R(1, \phi_c)) : c \in \mathbb{R}\}$.

(b) As n grows or $\sigma_0 \rightarrow 0$ the risk body expands out toward the boundary of the square $[0, 1]^2$; see the plots below.

(c) As $\theta_1 \rightarrow \theta_0 = 0$, the risk body contracts toward the diagonal line $(\alpha, 1 - \alpha)$ - since the testing problem becomes harder. See the plots below.

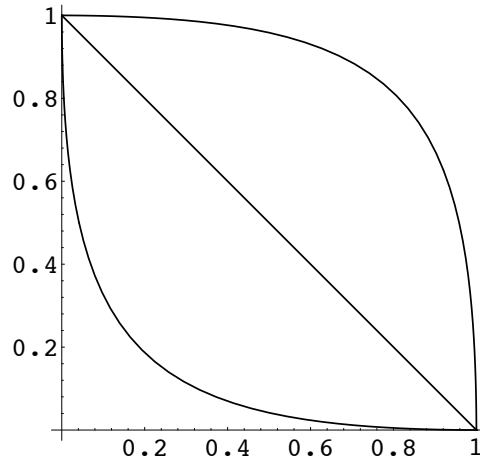


Figure 1: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = 1$.

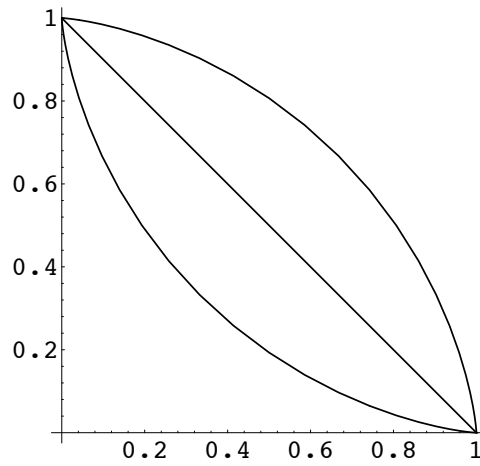


Figure 2: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = .5$.

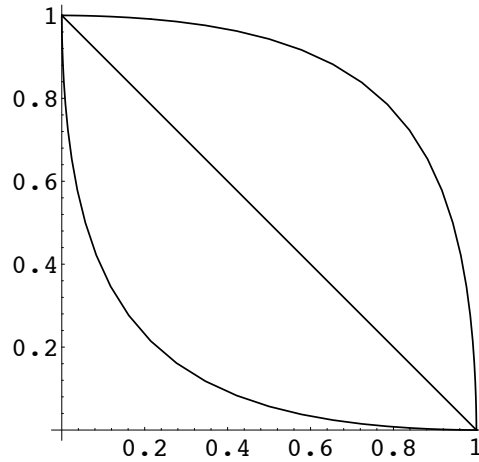


Figure 3: Risks for normal mean test, $n = 10$, $\sigma_0 = 1$, $\theta_1 = .5$.

5. Consider the family of densities

$$p_\alpha(x) = \frac{\sin(\pi\alpha)}{2\pi(\cosh(\alpha x) + \cos(\pi\alpha))},$$

where $\alpha \in (0, 1)$. (This family of densities was introduced in the context of binary mixed effects models by Wang and Louis (2003).)

(a) Compute $p_{1/2}(x)$ and compare it to the standard logistic density by writing the standard logistic density in terms of $\cosh(x/2)$.

(b) Show that the family of densities $\{p_\alpha\}$ is log-concave for $\alpha \in (0, 1/2]$, but that it fails to be log-concave for α in $(1/2, 1]$. Thus for each $\alpha \in (0, 1/2]$ the family $\{p_\alpha(\cdot - \theta) : \theta \in \mathbb{R}\}$ has monotone likelihood ratio.

(c) Unfortunately the result of (b) does not carry over to a sample of size n . If X_1, \dots, X_n are i.i.d. $P_\theta \equiv P_{\theta, \alpha}$ with density $p_\alpha(x - \theta)$ as in (b) then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

Solution: (a) When $\alpha = 1/2$, $\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$, and hence

$$p_{1/2}(x) = \frac{1}{2\pi \cosh(x/2)}.$$

On the other hand, the standard logistic density is

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2} = \frac{1}{e^x(1 + e^{-x})^2} = \frac{1}{(e^{x/2} + e^{-x/2})^2} = \frac{1}{4\{\cosh(x/2)\}^2}.$$

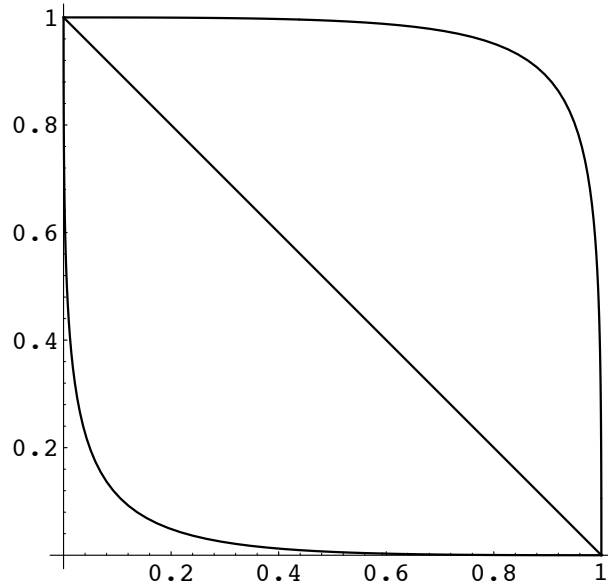


Figure 4: Risks for normal mean test, $n = 25$, $\sigma_0 = 1$, $\theta_1 = .5$.

Thus

$$\frac{f(x)}{p_{1/2}(x)} = \frac{\pi}{2} \frac{1}{\cosh(x/2)} \propto p_{1/2}(x) \rightarrow 0 \text{ as } x \rightarrow \infty,$$

so $p_{1/2}$ has slightly heavier tails than the standard logistic distribution.

(b) To see that p_α is log-concave for $\alpha \in (0, 1/2]$ we let $\varphi_\alpha(x) \equiv -\log p_\alpha(x)$ and compute

$$\begin{aligned} \varphi'_\alpha(x) &= \frac{\alpha \sinh(\alpha x)}{\cosh(\alpha x) + \cos(\alpha\pi)}, \\ \varphi''_\alpha(x) &= \frac{\alpha^2(1 + \cos(\alpha\pi) \cosh(\alpha x))}{(\cos(\alpha\pi) + \cosh(\alpha x))^2}. \end{aligned}$$

Since $\cos(\alpha\pi) \geq 0$ for $0 < \alpha \leq 1/2$, it follows that $\varphi''_\alpha(x) \geq 0$ for these values of α and hence p_α is log-concave for $\alpha \in (0, 1/2]$. Since $\cos(\alpha\pi) < 0$ for $0 < \alpha < 1$ and $\cosh(\alpha x)$ is unbounded, it follows that $\varphi''_\alpha(x) < 0$ for x sufficiently large, and hence p_α fails to be log-concave for $\alpha \in (1/2, 1)$. On the other hand, φ_α continues to be bounded below by some finite value even when $\alpha \in (1/2, 1)$; this is an example of a family that is *semi-log concave* in the terminology of Cattiaux and Guillin (2013).

(c) By the theory developed in Example 1.7 of the notes, the locally most powerful test of $H : \theta = \theta_0$ versus $K : \theta > \theta_0$ for X_1, \dots, X_n i.i.d. as $g(\cdot - \theta)$ with $g = p_\alpha$ is of the form

$$\phi(\underline{X}) = 1\{S_n(\theta_0) > k\}$$

where

$$S_n(\theta_0) = n^{-1/2} \sum_{i=1}^n -\frac{g'}{g}(X_i - \theta_0).$$

Taking $k = \sqrt{I_g z_\alpha}$ yields a test of approximate size α for n large. As noted in the solution to the optional problem below, I have succeeded in computing I_{p_α} only in the case $\alpha = 1/2$: then $I_{p_{1/2}} = 1/8$.

6. **Optional bonus problem 1.** (Subbotin densities) Let p_r denote the densities

$$p_r(x) = C_r \exp(-|x|^r/r)$$

for $x \in \mathbb{R}$ and $r > 0$ where $C_r = 1/[2\Gamma(1/r)r^{1/r-1}]$.

(a) Show that p_r is log-concave for all $r \geq 1$. Note that this family includes the Laplace (or double exponential) density for $r = 1$ and the Gaussian (or standard normal) density for $r = 2$.

(b) A density p is called *strongly log-concave* if $\varphi(x) \equiv -\log p(x)$ satisfies $\varphi''(x) \geq c > 0$ for some $c > 0$ and all $x \in \mathbb{R}$. Show that the only Subbotin density p_r that is strongly log-concave is p_2 , the standard Gaussian density.

Solution: (a) Note that $\varphi_r(x) \equiv -\log p_r(x) = r^{-1}|x|^r - \log C_r$ is a convex function for $r \geq 1$.

(b) Now $\varphi_r(x)$ defined in the solution to (a) is twice differentiable at all $x \neq 0$ with derivative $\varphi'_r(x) = \text{sign}(x)|x|^{r-1}$ and second derivative $\varphi''_r(x) = (r-1)|x|^{r-2}$. Note that $\varphi''_r(x)$ equals 0 when $r = 1$ and decreases to 0 as $x \rightarrow 0$ when $r > 2$ and as $x \rightarrow \pm\infty$ when $1 < r < 2$. When $r = 2$ $\varphi''_r(x)$ is bounded below by its (constant) value 1.

7. **Optional bonus problem 2.** Show that the Cauchy scale family of distributions given by

$$p_\theta(x) = \frac{1}{\pi\theta} \frac{1}{1 + (x/\theta)^2}$$

does not have monotone likelihood ratio, but that the distribution of the sufficient statistic $|X|$ (where $X \sim P_\theta$ with Cauchy density p_θ as in the last display) does have monotone likelihood ratio.

Solution: For $\theta' > \theta$ the ratio $p_{\theta'}/p_\theta$ is given by

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = \frac{\theta' \theta^2 + x^2}{\theta \theta'^2 + x^2}$$

which increases from $\theta/\theta' < 1$ to $\theta'/\theta > 1$ as x^2 increases from 0 to ∞ . Thus $\{p_\theta\}$ has MLR in $T(x) = x^2$ (or $|x|$).

Alternatively, the sufficient statistic $|X|$ has density

$$p_{\theta}^{|X|}(y) = 2p_{\theta}(y)1_{[0,\infty)}(y) = \frac{2}{\pi\theta} \frac{1}{1+(y/\theta)^2} 1\{y \geq 0\},$$

which has MLR in y .

8. **Optional bonus problem 3.** Can you prove anything about monotonicity of the power functions of the class of tests derived in problem 5(c)?

Solution: By the discussion in example 1.7 we see that for $g = p_{\alpha}$ as given in the current problem, the power functions of the locally most powerful test ϕ above will be monotone non-decreasing when $g = p_{\alpha}$ is log-concave, and hence for $\alpha \in (0, 1/2]$. I have succeeded in computing the Fisher information for location for $g = p_{\alpha}$ only when $\alpha = 1/2$: $I_{p_{1/2}} = 1/8$. I anticipate that it is very small for a near 0 and quite large for a near 1, but it seems to be somewhat tricky to compute numerically.