Statistics 582, Final Exam Solutions

Wellner; 3/18/2015

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A level α *permutation test* (in the setting of $m + n = N$ observations $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ i.i.d. some $F \in \mathcal{F}_c$.

   (b) A *Bayes decision rule* in a general decision problem with loss function $L(\theta, d)$.

   (c) A *uniformly most powerful size* α *test* of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A *uniformly most powerful invariant size* α *test* of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$ in a testing problem with a group G of transformations on the sample space $\mathcal{X}$.

   (e) An *inadmissible decision rule* d.

   Solution: See Course Notes, Chapters 5 and 6

2. (36 points) **State** *any three* of the following results:

   (a) The Neyman-Pearson lemma.

   (b) The Karlin-Rubin theorem.

   (c) A conditional limit theorem about the large sample behavior of posterior distributions.

   (d) A result concerning some optimality property of $\overline{X}_n$ in the context of estimating the mean μ when $X_1, \ldots, X_n$ are i.i.d. $N_d(\mu, \sigma^2 I)$ with σ^2 known and $d \geq 3$

   (e) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   Solution: See Course Notes, Chapters 5 and 6

Do either problem 3 or problem 4.

3. (50 points) Suppose that $p_\theta(x) = \theta^{-1} h(x/\theta)$, $\theta > 0$ where h is an even density function (i.e. symmetric about 0); this is a *scale family*. Suppose that $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ. For inference about θ we can reduce to $|X_1|, \ldots, |X_n|$ since these are sufficient for θ.

   (a) Show that the density q_θ of the $|X_i|$'s is $q_\theta(x) = 2\theta^{-1} h(x/\theta) 1_{(0,\infty)}(x)$ for $\theta > 0$.

   (b) Let $Y_i \equiv \log |X_i|$ for $i = 1, \ldots, n$, and let $\eta \equiv \log \theta$. Show that the density of the Y_i's is given by $\tilde{q}_\eta(y) = 2h(e^{y-\eta}) e^{y-\eta} \equiv g(y - \eta)$ for $y \in \mathbb{R}$ and $\eta \in \mathbb{R}$ where $g(z) \equiv 2h(e^z)e^z$. [Thus the log-transform converts a positive axis scale problem into a related location problem.]

   (c) Show that the scale family $\{q_\theta : \theta > 0\}$ has MLR (monotone likelihood ratio in x) if and only if $g(z)$ is log-concave; i.e. if and only if $-\log(g(z)) = -\log 2 - \log h(e^z) + z$ is a convex function of z; or equivalently if and only if $-\log h(e^z)$ is a convex function of z.

   (d) Apply (a)-(c) to the Cauchy scale family $p_\theta(x) = \theta^{-1} h(x/\theta)$ with $h(x) = \pi^{-1}(1 + x^2)^{-1}$.

Solution: This is from Lehmann and Romano, TSH, page 324, Example 8.2.2. The upshot is that the MLR property is more restrictive for location than for scale.

(a) If P_θ has density p_θ, then it follows that $Q_\theta(x) = P_\theta(|X_i| \le x) = P_\theta(X_i \le x) - P_\theta(X_i \le -x)$ has density $q_\theta(x) = p_\theta(x) + p_\theta(-x) = 2\theta^{-1} h(x/\theta)$ for $x > 0$ since p_θ is symmetric about 0.

(b) Now $\tilde{Q}_\theta(y) = P(\log |X_i| \le y) = Q_\theta(|X_i| \le e^y)$, so

$$\tilde{q}_\eta(y) = q_\theta(e^y) e^y = 2h(e^y/\theta) e^y/\theta = 2h(e^{y-\eta}) e^{y-\eta}.$$

(c) Let $0 < \theta < \theta'$. Then $q_{\theta'}(x)/q_\theta(x) = (\theta/\theta') h(x/\theta')/h(x/\theta)$ is a monotone increasing function of x if and only if $h(x/\theta')/h(x/\theta)$ is a monotone increasing function of x, and this holds if and only if $h(e^{y-\eta'})/h(e^{y-\eta})$ is a monotone increasing function of y, and this holds if and only if $\tilde{q}_{\eta'}(y)/\tilde{q}_\eta(y)$ is a monotone non-decreasing function of y. Since $\tilde{q}_\eta(y) = g(y - \eta)$ is a location family, we know that this has MLR if and only if g is log-concave. But since $g(z) = 2h(e^z)e^z$, g is log-concave if and only if $\psi(z) \equiv -\log 2h(e^z)e^z = -\log 2 - z - \log h(e^z)$ is a convex function. Since $-\log 2 - z$ is linear, this holds if and only if $-\log h(e^z)$ is convex.

(d) When $h(x) = \pi^{-1}(1 + x^2)^{-1}$, $\psi(z) \equiv -\log h(e^z) = \log(\pi) + \log(1 + e^{2z})$. But then $\psi'(z) = 2e^{2z}/(1 + e^{2z}) = 2 - 2/(1 + e^{2z})$, and hence

$$\psi''(z) = \frac{4e^{2z}}{(1 + e^{2z})^2} \ge 0.$$

Thus $-\log h(e^z)$ is convex, and the Cauchy scale family has MLR.

4. (50 points) Suppose that an urn contains N balls with the numbers $a_N(1), \ldots, a_N(N)$ written on the balls. Suppose that a sample of n balls is drawn from the urn without replacement: let the numbers on the sampled balls be $Y_1, \ldots, Y_n$, and let $\overline{Y}_n = n^{-1}\sum_{i=1}^n Y_i$.
(a) What is the mean of $\overline{Y}_n$?
(b) What is the variance of $\overline{Y}_n$?
(c) If $\underline{R} = (R_1, \ldots, R_N)$ is a random permutation of $\{1, \ldots, N\}$, what is the relationship between $(Y_1, \ldots, Y_n)$ and $(a_N(R_1), \ldots, a_N(R_n))$?
(d) Under some condition on the numbers $a_N(i)$, a CLT holds for an appropriately standardized version of $\overline{Y}_n$, and hence also for $\sum_{j=1}^n a_N(R_j)$. State this condition and the theorem.
(e) If $a_N(j) = 1_{\{1,\ldots,D\}}(j)$, for $j = 1, \ldots, N$ and some integer $D \in \{1, \ldots N-1\}$, compute the mean and variance in (a) and (b), and make the conclusion of the CLT in (d) explicit. What is the name of the distribution of $\sum_{i=1}^n Y_i$ in this case?

Solution: (a) $E(\overline{Y}_n) = \overline{a}_N \equiv N^{-1}\sum_{i=1}^N a_N(i)$.
(b) $Var(\overline{Y}_n) = n^{-1}\sigma_a^2(1 - \frac{n-1}{N_1}) \equiv \sigma_N^2$ where $\sigma_a^2 = N^{-1}\sum_{i=1}^N (a_N(i) - \overline{a}_N)^2$.
(c) The random vectors $\underline{Y} = (Y_1, \ldots, Y_n)$ and $(a_N(R_1), \ldots, a_N(R_n))$ satisfy $(Y_1, \ldots, Y_n) \stackrel{d}{=} (a_N(R_1), \ldots, a_N(R_n))$.
(d) If $0 < \liminf(n/N) \le \limsup(n/N) < 1$, then with σ_N^2 as in (b)

$$\frac{\overline{Y}_n - \overline{a}_N}{\sigma_N} \rightarrow_d Z \sim N(0,1)$$

if and only if

$$\eta_N \equiv \frac{\max_{1\le i\le N}(a_N(i) - \overline{a}_N)^2}{\sum_{i=1}^N (a_N(i) - \overline{a}_N)^2} \rightarrow 0. \tag{1}$$

(e) When $a_N(j) = 1_{\{1,\ldots,D\}}(j)$, we have $\overline{a}_N = D/N$, and $N^{-1}\sum_{i=1}^N a_N(i)^2 = D/N$, so $\sigma_a^2 = (D/N) - (D/N)^2 = (D/N)(1 - D/N)$, and the Noether condition (1) holds since

$$\eta_N = \frac{\max\{(1 - D/N)^2, (D/N)^2\}}{N\sigma_a^2} = \frac{\max\{(1 - D/N)^2, (D/N)^2\}}{D(1 - D/N)} \rightarrow 0$$

if $D = D_N \rightarrow \infty$ and $\limsup(D/N) < 1$. In this case $\sum_{i=1}^N Y_i$ is the number of Red balls in a sample of size n drawn without replacement from an urn containing

D red balls and $N - D$ white balls, and thus has a Hypergeometric(n, D, N) distribution given by

$$P(\sum_{i=1}^{n} Y_i = k) = \frac{\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}, \quad k \in \{0 \vee (n - N + D) \ldots, D \wedge n\}.$$

Do either problem 5 or problem 6.

5. (48 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. exponential(θ) random variables so that $1 - F_\theta(x) = \exp(-\theta x)$ for $x \geq 0$ and $\theta > 0$.
(a) Find the MLE of θ.
(b) If $\boldsymbol{\theta} \sim$ Gamma(α, β) so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta)$$

and $E(\boldsymbol{\theta}) = \alpha/\beta$, find the posterior distribution of $\boldsymbol{\theta}$.
(c) Find the Bayes estimator of θ for squared error loss. Is it consistent?
(d) What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?

Solution: (a) The joint density of the data is

$$p(\underline{x}; \theta) = \prod_{i=1}^{n} \theta \exp(-\theta x_i) = \theta^n \exp(-\theta \sum_1^n x_i)$$

so $\log p(\underline{X}; \theta) = n \log \theta - \theta \sum_1^n X_i$, and we see that this is maximized by $\hat{\theta}_n = 1/\overline{X}$.
(b) The joint density of the data and θ with the Gamma prior is

$$p(\underline{x}; \theta)\lambda(\theta) \propto \theta^n \exp(-\theta \sum_1^n x_i)\theta^{\alpha-1} \exp(-\beta\theta) = \theta^{\alpha+n-1} \exp(-(\beta + \sum_1^n x_i)\theta),$$

so it follows that $(\theta|\underline{X}) \sim$ Gamma$(\alpha + n, \beta + \sum_1^n X_i)$.
(c) The Bayes estimator of θ is

$$\begin{aligned} E(\theta|\underline{X}) &= (\alpha + n)/(\beta + \sum_1^n X_i) \\ &= \frac{1 + \alpha/n}{\overline{X} + \beta/n} \rightarrow_{a.s.} \frac{1 + 0}{1/\theta + 0} = \theta_0 \end{aligned}$$

(d) Now

$$\begin{aligned}
\sqrt{n}(\theta - E(\theta|\underline{X})) &\stackrel{d}{=} \sqrt{n}\left(\text{Gamma}(\alpha+n, \beta+\sum_1^n X_i) - \frac{\alpha+n}{\beta+\sum_1^n X_i}\right) \\
&\stackrel{d}{=} \sqrt{n}\frac{1}{\beta+\sum_1^n X_i}\left(Y_0+Y_1+\cdots+Y_n-(\alpha+n)\right) \\
&= \frac{n}{\beta+\sum_1^n X_i}\frac{1}{\sqrt{n}}((Y_0-\alpha)+(Y_1-1)+\cdots+(Y_n-1)) \\
&\to_d \frac{1}{1/\theta_0}Z \sim N(0,\theta_0^2) = N(0, I^{-1}(\theta_0) \qquad \text{a.s.}
\end{aligned}$$

where the X_i are i.i.d. Exponential(θ_0). Here $Y_0 \sim \text{Gamma}(\alpha, 1)$ is independent of $Y_1, \ldots, Y_n$ which are i.i.d. Gamma$(1,1)$ random variables.

6. (48 points) Suppose that conditional on $\boldsymbol{\theta} = \theta$, $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$, and suppose that $\boldsymbol{\theta} \sim N(\mu, \tau^2)$.

(a) Find the posterior distribution of $\boldsymbol{\theta}$ given $\underline{X} \equiv (X_1, \ldots, X_n)$.

(b) Find the marginal distribution of $\underline{X}$.

(c) Find the Bayes rule $d_B(\underline{X})$ for estimation of θ with squared error loss.

(d) If θ_0 is the true value of θ (so that $X_1, \ldots, X_n$ are i.i.d. $N(\theta_0, \sigma^2)$), show that the Bayes rule $d_B(\underline{X})$ in (c) is biased but consistent.

(e) If θ_0 is the true value of θ, show that $\sqrt{n}(d_B(\underline{X}) - \theta_0) \to_d$ something and identify "something".

(f) If θ_0 is the true value of θ, find the distribution of $(\sqrt{n}(\boldsymbol{\theta} - \overline{X}_n)|\underline{X})$ explicitly and show that $(\sqrt{n}(\boldsymbol{\theta} - \overline{X}_n)|\underline{X}) \to_d N(0, \sigma^2)$ with probability 1.

Solution: (a) We can reduce by sufficiency to $\overline{X}_n \sim N(\theta, \sigma^2/n)$. If $(X|\boldsymbol{\theta} = \theta) \sim N(\theta, \sigma^2)$, $\boldsymbol{\theta} \sim N(\mu, \sigma^2)$, then

$$(\boldsymbol{\theta}|\underline{X}) \sim N\left(\frac{1/\tau^2}{1/\tau^2 + n/\sigma^2}\mu + \frac{n/\sigma^2}{1/\tau^2 + n/\sigma^2}\overline{X}_n, \frac{1}{1/\tau^2 + n/\sigma^2}\right).$$

(b) Let $\underline{Z} \equiv \underline{X} - \theta\underline{1} \sim N_n(0, \sigma^2 I)$. Then the marginal distribution of $\underline{X}$ is that of $\underline{Z} + \boldsymbol{\theta}\underline{1}$ where $\underline{Z}$ and $\boldsymbol{\theta}$ are independent. Thus $E(\underline{X} = \mu\underline{1}$ and $Cov(\underline{X}) = \sigma^2 I + \tau^2 \underline{1}\underline{1}^T$, and the marginal distribution of $\underline{X}$ is $N_n(\mu\underline{1}, \sigma^2 I + \tau^2\underline{1}\underline{1}^T)$.
(c) From (a) it follows that for squared error loss the Bayes rule is

$$d_B(\underline{X}) = E(\boldsymbol{\theta}|\underline{X}) = \frac{1/\tau^2}{1/\tau^2 + n/\sigma^2}\mu + \frac{n/\sigma^2}{1/\tau^2 + n/\sigma^2}\overline{X}_n.$$

(d) If θ_0 is the true value of θ, it follows from (c) that

$$\begin{aligned} E(d_B(\underline{X})) &= \frac{1/\tau^2}{1/\tau^2 + n/\sigma^2}\mu + \frac{n/\sigma^2}{1/\tau^2 + n/\sigma^2}\theta_0 \\ &= p_n\mu + (1-p_n)\theta_0 \neq \theta_0 \end{aligned}$$

so d_B is biased, but $d_B(\underline{X}) \rightarrow_{a.s.} 0 \cdot \mu + 1 \cdot \theta_0 = \theta_0$, so d_B is a consistent estimator of θ_0.

(e) If θ_0 is the true value of θ, it follows from (c) that with $Z \sim N(0,1)$ and p_n as defined in (d),

$$\begin{aligned} \sqrt{n}(d_B(\underline{X}) - \theta_0) &= \sqrt{n}(p_n\mu + (1-p_n)\overline{X}_n - \theta_0) \\ &= \sqrt{n}(p_n\mu + (1-p_n)\overline{X}_n - p_n\theta_0 - (1-p_n)\theta_0) \\ &= \sqrt{n}p_n(\mu - \theta_0) + (1-p_n)\sqrt{n}(\overline{X}_n - \theta_0) \\ &\rightarrow_d 0 \cdot (\mu - \theta_0) + 1 \cdot \sigma Z = \sigma Z \sim N(0, \sigma^2) \end{aligned}$$

since $p_n \rightarrow 0$, $\sqrt{n}p_n \rightarrow 0$, and $\sqrt{n}(\overline{X}_n - \theta_0) \stackrel{d}{=} \sigma Z$.

(f) If θ_0 is the true value of θ, then

$$(\sqrt{n}(\boldsymbol{\theta} - \overline{X}_n)|\underline{X}) \sim N(\sqrt{n}p_n(\mu - \overline{X}_n), \tau^2 n p_n).$$

Now $\sqrt{n}p_n(\mu - \overline{X}_n) \rightarrow_{a.s.} 0 \cdot (\mu - \theta_0) = 0$, and $\tau^2 n p_n \rightarrow \sigma^2$. Thus the posterior distributions converge to $N(0, \sigma^2)$ with probability 1.

Do **either** problem 7 **or** problem 8.

7. (48 points) Suppose that $X_i \sim \text{Poisson}(\mu i)$, $i = 1, \ldots, m$ are independent, and that $Y_j \sim \text{Poisson}(\nu j)$, $j = 1, \ldots, n$ are also independent and independent of the X_i's. Consider testing $H : \mu \geq \nu$ versus $K : \mu < \nu$.

   (a) Show that we can reduce by sufficiency to $R \equiv \sum_{i=1}^m X_i$ and $S \equiv \sum_{j=1}^n Y_j$.

   (b) What are the distributions of R and S?

   (c) Find a level α UMP - unbiased test of H versus K and indicate exactly how to carry it out, identifying the relevant conditional distribution explicitly.

   (d) Describe the Bayes test of H versus K (for $0-1$ loss assuming a prior distribution Λ of (μ, ν). Can you describe the rejection region of this test when Λ is the product of two independent Γ priors (i.e. $\mu \sim \Gamma(\alpha, \beta)$ and $\nu \sim \Gamma(\gamma, \delta)$)?

Solution: (a) Note that the joint mass function of the X_i's is given by

$$\begin{aligned} p_\mu(\underline{x}) &= \prod_{i=1}^m \exp(-i\mu)\frac{(i\mu)^{x_i}}{x_i!} = \prod_{i=1}^m \exp(-i\mu)\exp(x_i\log(i\mu))\frac{1}{x_i!} \\ &= \exp(-\mu\sum_{i=1}^m i + \sum_{i=1}^m x_i \log\mu + \sum_{i=1}^m x_i \log i)\frac{1}{\prod_{i=1}^m x_i!} \\ &= c(\mu)\exp(\sum_1^m x_i(\log\mu))h(\underline{x}), \end{aligned}$$

and hence $R = \sum_{i=1}^m X_i$ is sufficient for μ. Similarly, $S = \sum_{j=1}^n Y_j$ is sufficient for ν.

(b) Since sums of independent Poisson random variables are again poisson with intensity parameter the sum of the intensity parameters of the summands, we see that

$$R = \sum_{i=1}^m X_i \sim \text{Poisson}(\mu\sum_1^m i) = \text{Poisson}(\mu m(m+1)/2) = \text{Poisson}(a\mu)$$

where $a \equiv m(m+1)/2$. Similarly, $S = \sum_{j=1}^n Y_j \sim \text{Poisson}(b\nu)$ where $b \equiv n(n+1)/2$.

(c) The joint mass function of R and S is

$$\begin{aligned} p(r,s;\mu,\nu) &= e^{-a\mu}\frac{(a\mu)^r}{r!}e^{-b\nu}\frac{(b\nu)^s}{s!} \\ &= c(\mu,\nu)\exp(r\log(a\mu) + s\log(b\nu))h(r,s) \\ &= c(\mu,\nu)\exp(r\log(\mu) + s\log(\nu))\tilde{h}(r,s) \\ &= c(\mu,\nu)\exp(s(\log\nu - \log\mu) + s\log\mu + r\log\mu)\tilde{h}(r,s) \\ &= c(\mu,\nu)\exp(\theta s + \eta(r+s))\tilde{h}(r,s) \end{aligned}$$

where $\theta \equiv \log(\nu/\mu)$, $\eta \equiv \log\mu$, and we see that $T \equiv R + S$ is sufficient for $\Theta_B \equiv \overline{\Theta}_0 \cap \overline{\Theta}_1$, and testing H versus K is equivalent to testing $\theta \le 0$ versus $\theta > 0$. Thus the UMP unbiased test of H versus K is of the form

$$\phi(S) = 1\{S > c_T\} + \gamma_T 1\{S = c_T\}.$$

where c_T and γ_T are determined so that $E\{\phi(S)|T\} = \alpha$. But

$$(S|T) \sim \text{Binomial}\left(T, \frac{\nu b}{\mu a + \nu b}\right) = \text{Binomial}\left(T, \frac{b}{a+b}\right)$$

under $\mu = \nu$.

(d) When (μ,ν) have the prior distribution Λ, the Bayes test of H versus K

is "reject H if $P(\nu > \mu|R,S) > P(\nu \le \mu|R,S)$", or, equivalently, if $P(\nu > \mu|R,S) > 1/2$. When Λ has density $\lambda(\mu,\nu)$ given by the product of two Gamma densities as described, then

$$\begin{aligned} p_\mu(r)\lambda_{\alpha,\beta}(\mu) &= \exp(-a\mu)\frac{(a\mu)^r}{r!}\cdot\frac{\beta(\beta\mu)^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta\mu} \\ &\propto \mu^{\alpha+r-1}\exp(-(a+\beta)\mu), \end{aligned}$$

so we see that the posterior density of μ given $R=r$ is Gamma$(\alpha+r, a+\beta)$. Similarly, the posterior density of ν given $S=s$ is Gamma$(\gamma+s, b+\delta)$. Thus we find that the posterior probability in question is, with $a = m(m+1)/2$, $b = n(n+1)/2$,

$$P(\nu > \mu|R,S) = P(\text{Gamma}(\gamma+S, b+\delta) > \text{Gamma}(\alpha+R, a+\beta))$$

where the two Gamma random variables are independent.

8. (48 points) Suppose that

$$Y_i \sim N(\mu + \Delta(i-(n+1)/2), \sigma^2) \equiv N(\mu+\Delta x_i, \sigma^2), \quad i=1,\ldots,n,$$

are independent with $\Delta \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma^2 > 0$.

(a) Consider testing $H: \Delta = 0$ versus $K: \ \Delta \ne 0$. Relate this testing problem to our theory of the general linear model in Example 6.3.15 of the course notes. What is the subspace $\mathbf{L}$? What is the subspace $\mathbf{L}_1$? Identify k and r explicitly. Find the least squares estimators of μ and Δ under both the larger model and the smaller model specified by the null hypothesis.

(b) Find the UMP - G invariant test of H versus K (where G is the group of transformations discussed in Example 3.15). Describe the distribution of your test statistic under the general model.

(c) If the hypotheses are changed to $H: \Delta \le 0$ versus $K: \ \Delta > 0$, find the UMP unbiased level α test of H versus K. Can you carry out the test unconditionally?

(d) Describe the power function of the test in (c).

Solution: This is a special case of the third example (Example 3.18) in the handout in class on 13 March, but with X_i renamed as Y_i, $z_i = i-(n+1)2$, β_0 called μ, and β_1 called Δ here.
(a) The subspace $\mathbf{L}$ of $\mathbb{R}^n$ is spanned by the two vectors $\underline{1}$ and $\underline{z}$ where $\underline{z}$ has components $z_i = i-(n+1)/2$, so $k=2$. The subspace $\mathbf{L}_1$ is the subspace spanned by the single vector $\underline{1}$, and hence $r=1$. The least squares estimators

under the larger model are $\hat{\mu} = \overline{Y}_n$ and $\hat{\Delta}_n = \sum_{i=1}^n z_i Y_i / \sum_{i=1}^n z_i^2$. The least squares estimator under the null hypothesis is just $\hat{\hat{\mu}} = \overline{Y}_n$.
(b) The UMP G-invariant level α test of $H : \Delta = 0$ versus $K : \Delta \neq 0$ is given by "reject H if $F > F_{1,n-2,\alpha}$" where

$$F = \frac{\sum_{i=1}^n (\widehat{\Delta} z_i)^2}{\sum_{i=1}^n (Y_i - \overline{Y} - \widehat{\Delta} z_i)^2/(n-2)}$$

which has an $F_{1,n-2}(\delta^2)$ distribution under the general hypothesis where $\delta^2 = \Delta^2 \sum_1^n z_i^2 / \sigma^2$.
(c) To find the UMP unbiased test of $H : \Delta \leq 0$ versus $K : \Delta > 0$, we first consider the joint density of the observations: this is given by

$$\begin{aligned} p_{\mu,\Delta}(\underline{y}) &= (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu - \Delta z_i)^2 \right) \\ &= (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2} \sum_1^n y_i^2 + \frac{1}{\sigma^2} \sum_1^n (\mu + \Delta z_i) y_i - \frac{\sum_1^n (\mu + \Delta z_i)^2}{2\sigma^2} \right) \\ &= C(\mu, \Delta, \sigma^2) \exp\left(\frac{\Delta}{\sigma^2} \sum_1^n z_i y_i + \frac{\mu}{\sigma^2} \sum_1^n y_i - \frac{1}{\sigma^2} \sum_1^n y_i^2 \right). \end{aligned}$$

Thus we see that under $\Theta_B = \{(\mu, 0, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 \geq 0\}$ we have $\underline{T} = (T_1, T_2) \equiv (\sum_1^n Y_i, \sum_1^n Y_i^2)$ is sufficient and complete for $\underline{\eta} = (\eta_1, \eta_2) \equiv (\mu/\sigma^2, -1/(2\sigma^2))$, and the level α UMP unbiased test of H versus K rejects H if $U(\underline{Y}) \equiv \sum_1^n z_i Y_i > c(\underline{T})$ where $c(\underline{T})$ is determined by $P_{\Theta_B}(U(\underline{Y}) > c(\underline{T}) | \underline{T}) = \alpha$. But note that $\hat{\Delta} = U / \sum_1^n z_i^2 = \sum_1^n z_i Y_i / \sum_1^n z_i^2$ is a monotone increasing function of U, and

$$\begin{aligned} \tau &\equiv \frac{\sqrt{\sum_1^n z_i^2}\,\widehat{\Delta}}{\sqrt{\sum_{i=1}^n (Y_i - \overline{Y} - \widehat{\Delta} z_i)^2/(n-2)}} \\ &= \frac{\sqrt{(n-2)\sum_1^n z_i^2}\,\widehat{\Delta}}{\sqrt{\sum_{i=1}^n Y_i^2 - n\overline{Y}^2 - \widehat{\Delta}^2 \sum_1^n z_i^2)}} \end{aligned}$$

is a monotone increasing function of U (and $\widehat{\Delta}$) for fixed values of $\underline{T} = (\sum_1^n Y_i, \sum_1^n Y_i^2)$. Moreover, τ is ancillary under $\Delta = 0$ (with a t_{n-2} distribution), and hence is independent of $\underline{T}$ under $\Delta = 0$ by Basu's theorem. Thus the UMP unbiased test of H versus K can be carried out unconditionally as "reject H if $\tau > t_{n-2,\alpha}$".