# Statistics 582, Problem Set 8

Wellner; 2/25/2015

**Reading:** Chapter 6, sections 6.2 and 6.3 (through page 34).
**Due:** Wednesday, March 4, 2015.
**Reminder:** Final Exam, Monday, March 16, 8:30 - 10:30 (MGH 271)

1. Consider the Locally Most Powerful test $\phi$ for testing $H : \theta \le 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.
(a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.
(b) What is the behavior of each of these two approximations for large values of $\theta$? Which of them shows that the power function decreases to 0 as $\theta \to \infty$? Why?

2. This is a continuation of the previous problem.
(a) Suppose you decide to base a test of the hypotheses in the setting of Example 6.1.5 on the sample mean $\overline{X}_n$. Show how to carry out this test at exact size $0 < \alpha < 1/2$, and compute its power function $\beta_\phi(\theta)$. [Hint: Recall that for $\theta_0 = 0$, and Cauchy data, $\overline{X}_n \stackrel{d}{=} X_1$.]
(b) Alternatively, consider a test of the same hypotheses on the sample median $\mathbb{F}_n^{-1}(1/2)$, or perhaps more simply, the proportion of observations exceeding $\theta_0 = 0$. Show how to carry out this test at exact and approximate size $\alpha \in (0, 1/2)$. Compute the power function of your new test and show that it has a monotone non-decreasing power function $\beta_\phi(\theta)$.
(c) Compare the power functions (or best approximations thereof) for the tests in (a) and (b) and problem 1.

3. Let $X$ and $Y$ be random variables with joint density

$$p_{X,Y}(x, y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y).$$

(a) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \le \mu + 1$ versus $H_1 : \lambda > \mu + 1$.
(b) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda = \mu$ versus $H_1 : \lambda \ne \mu$.
(c) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \ge 2\mu$ versus $H_1 : \lambda < 2\mu$.

(d) What happens when $X_1, \ldots, X_m$ are i.i.d. Exponential($\lambda$) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential($\mu$)?

4. Lehmann and Romano, TSH, problem 4.3, page 139, modified: Let $X \sim$ Binomial$(n, p)$, and consider testing $H : p = p_0$ versus $K : p \neq p_0$ at level $\alpha = \alpha$. Determine the boundary values of the UMP unbiased test for $n = 20$ with $\alpha = .05$, $p_0 = .2$, and with $\alpha = .10$, $p_0 = .4$. In each case plot the power functions of both the unbiased and the equal-tails test.

5. **Optional bonus problem 1:** Let $\mathcal{P} = \{p_\theta : \ \theta \in \Theta\}$ where $p_\theta$ is a family of densities with respect to a fixed dominating measure $\mu$ defined on a sample space $\mathcal{X}$.
(a) Suppose that the densities $p_\theta(x) \equiv p(x, \theta)$ have a second mixed partial derivative and that
$$\frac{\partial}{\partial x}\frac{\partial}{\partial \theta} \log p(x, \theta) \geq 0$$
for all $x \in \mathbb{R}$ and $\theta \in \Theta$. Show that the inequality in the last display implies that $\mathcal{P}$ has monotone likelihood ratio. [Hint: use the fundamental theorem of calculus twice.] (b) Show that the condition in (a) is equivalent to
$$p(x, \theta)\frac{\partial}{\partial x}\frac{\partial}{\partial \theta}p(x, \theta) \geq \frac{\partial}{\partial \theta}p(x, \theta)\frac{\partial}{\partial x}p(x, \theta) \quad \text{for all} \quad \theta, \ x.$$

6. **Optional bonus problem 2:** Let $X_1, \ldots, X_n$ be a sample of size $n$ from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
(a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.
(b) Find the subclass of the tests that are independent of $\theta_1$. These are UMP tests of $H_0$ versus $H_1' : \theta > \theta_0$.
(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is UMP of size $\alpha$ for testing $H_0' : \theta \leq \theta_0$ versus $H_1' : \theta > \theta_0$ but that $\phi$ is not admissible.
(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \leq b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size $\alpha$ for testing $H_0 : \theta = \theta_0$ versus $\theta \neq \theta_0$.

7. **Optional bonus problem 3:** let $X$ and $Y$ be independent random variables with geometric distributions
$$p_{X,Y}(x, y|\theta_1, \theta_2) = (1 - \theta_1)(1 - \theta_2)\theta_1^x \theta_2^y, \qquad x, y \in \{0, 1, \ldots\}.$$
where $0 < \theta_j < 1$, $j = 1, 2$. Find a UMP unbiased test of size $\alpha = .20$ for testing
(a) $H_0 : \theta_1 \leq \theta_2$ versus $H_1 : \theta_1 > \theta_2$.
(b) $H_0 : \theta_1 = \theta_2$ versus $H_1 : \ \theta_1 \neq \theta_2$.
(c) For what functions $\varphi(\theta_1, \theta_2)$ do our methods guarantee existence of a UMP unbiased test of $H_0 : \varphi(\theta_1, \theta_2) = 0$ versus $H_1 : \varphi(\theta_1, \theta_2) \neq 0$?