Statistics 582, Problem Set 7, revised

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Reading: Chapter 6, sections 6.1 and 6.2 (through page 19);
Lehmann and Romano, Chapter 3.

Due: Wednesday, February 25, 2015.

1. A random variable X takes on the values $1,\ 2,\ 3,\ 4$ with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.54	.08	.12	.26
$p_1(x)$	.22	.16	.36	.26

(a) Find a most powerful test of size $\alpha = .15$ for testing p_0 versus p_1 and determine its power.
(b) Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1-\phi)$.

2. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \ P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

3. (Problem 3.7, Lehmann and Romano, TSH, page 94, modified) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	$\theta/2$	θ	$.9-\theta$	$.1-\theta/2$

where $0 < \theta < .2$. For testing $H : \theta = .05$ against $\theta > .05$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = .5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(iii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
(a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
(b) What happens to the risk body as n grows or as $\sigma_0 \to 0$?
(c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
(d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : \ n \geq 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?

5. Consider the family of densities

$$p_\alpha(x) = \frac{\sin(\pi\alpha)}{2\pi(\cosh(\alpha x) + \cos(\pi\alpha))},$$

where $\alpha \in (0, 1/2]$. (This family of densities was introduced in the context of binary mixed effects models by Wang and Louis (2003).)
(a) Compute $p_{1/2}(x)$ and compare it to the standard logistic density by writing the standard logistic density in terms of $\cosh(x/2)$.
(b) Show that the family of densities $\{p_\alpha\}$ is log-concave for $\alpha \in (0, 1/2]$, but that it fails to be log-concave for α in $(1/2, 1]$. Thus for each $\alpha \in (0, 1/2]$ the family $\{p_\alpha(\cdot - \theta) : \ \theta \in \mathbb{R}\}$ has monotone likelihood ratio.
(c) Unfortunately the result of (b) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. $P_\theta \equiv P_{\theta,\alpha}$ with density $p_\alpha(x - \theta)$ as in (b) then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

6. **Optional bonus problem 1.** (Subbotin densities) Let p_r denote the densities

$$p_r(x) = C_r \exp(-|x|^r/r)$$

for $x \in \mathbb{R}$ and $r > 0$ where $C_r = 1/[2\Gamma(1/r)r^{1/r-1}]$.
(a) Show that p_r is log-concave for all $r \geq 1$. Note that this family includes the Laplace (or double exponential) density for $r = 1$ and the Gaussian (or standard normal) density for $r = 2$.
(b) A density p is called *strongly log-concave* if $\varphi(x) \equiv -\log p(x)$ satisfies $\varphi''(x) \geq c > 0$ for some $c > 0$ and all $x \in \mathbb{R}$. Show that the only Subbotin density p_r that is log-concave is p_2, the standard Gaussian density.

7. **Optional bonus problem 2.** Show that the Cauchy scale family of distributions given by

$$p_\theta(x) = \frac{1}{\pi\theta} \frac{1}{1 + (x/\theta)^2}$$

does not have monotone likelihood ratio, but that the distribution of the sufficient statistic $|X|$ (where $X \sim P_\theta$ with Cauchy density p_θ as in the last display) does have monotone likelihood ratio.

8. **Optional bonus problem 3.** Can you prove anything about monotonicity of the power functions of the class of tests derived in problem 5(c)?