Statistics 582, Problem Set 3

Wellner; 1/21/2015

Reading: Handout on Huber's $Z-$theorem, pages 1-10;
Chapter 5, sections 1-4.

Due: Wednesday, January 28, 2015.
Reminder: Make up lecture 1, 9:30 - 10:20, Wednesday, 28 January, SIG 226

1. Consider the zero-inflated Poisson distribution p_θ as described in Example 3 of the handout on M- and Z- theorems. Suppose that $X_1, \ldots, X_n$ i.i.d. p_θ are observed.
(a) Set up alternative estimating equations for $\theta = (\gamma, \lambda)$ where $\gamma \in [0,1]$ and $\lambda > 0$ based on $g_1(x) = x$ and $g_2(x) = x^2$. Express your alternative estimator $\hat{\theta}_n = (\hat{\gamma}_n, \hat{\lambda}_n)$ of θ explicitly in terms of the first and second moments, $\overline{X}_n$ and $\overline{X^2}_n$, of the data, and show that your estimators are consistent when the model holds.
(b) Use Huber's Z-theorem to show that $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N_2(0, \Sigma)$ and give the form of Σ.
(c) What happens if the X_i's are i.i.d. $p \notin \mathcal{P} = \{p_\theta : \theta \in \Theta\}$? Describe the parameter $\theta(P)$ to which $\hat{\theta}_n$ converges in probability and use Huber's theorem to establish a limit theorem for $\sqrt{n}(\hat{\theta}_n - \theta(P))$ in this case. What other methods do you have available in this case?

2. Let $\Theta = \{0, 1\} = \mathbf{A}$ where $0 =$ a patient has tuberculosis, $1 =$ a patient does not have tuberculosis. Let X be the number of positive reactions to two different tuberculosis tests, so that $\mathbf{X} = \{0, 1, 2\}$, and suppose that X has the following distributions

x	0	1	2
$p_0(x)$	.02	.13	.85
$p_1(x)$	.70	.27	.03

If the losses are given by $L(1,1) = L(0,0) = 0$, $L(0,1) = 100$, $L(1,0) = 10$, and the prior $\lambda = (\lambda_0, \lambda_1) = (.2, .8)$, find the Bayes rule d_B and the minimax rule d_M. Plot the risk set and label the non-randomized decision rules.

3. Consider testing the simple hypothesis $H_0 : X \sim P_0$ versus the simple alternative $H_1 : X \sim P_1$. Let ϕ be a test of H_0 versus H_1, and let $a \equiv E_1(1 - \phi)$, $b \equiv E_0 \phi$.
(a) Find a test ϕ which minimizes $a + Db$ where D is a fixed number.
(b) When $D = 1$, relate the minimized total $a + b$ to the risk and to the total variation distance $d_{TV}(P_0, P_1)$ between P_0 and P_1 (or $\int p_0 \wedge p_1 d\mu$ for a dominating measure μ, e.g. $P_0 + P_1$).
(c) Carry the computations of (b) through in the context of problem 1 when the losses are $L(0,0) = L(1,1) = 0$, $L(0,1) = 10 = L(1,0)$, and the prior distribution is $\lambda = (\lambda_0, \lambda_1) = (.3, .7)$.

4. Let $\mathcal{X} = \{0, 1\}$, $\mathcal{A} = \Theta = \{1, 2\}$, and assume that the losses are given by $L(1,1) = L(2,2) = 0$, $L(1,2) = a$, $L(2,1) = b$. Suppose that the statistician can observe

either X or Y where

$$p_1(1) = P_1(X=1) = 2/3, \qquad p_2(1) = P_2(X=1) = 1/2,$$
$$p_1^*(1) = P_1(Y=1) = 3/4, \qquad p_2^*(1) = P_2(Y=1) = 1/2.$$

Let $\underset{\sim}{\lambda} = (\lambda, 1-\lambda)$, $\lambda \in [0,1]$ be the prior distribution over Θ.
(a) Find the Bayes risk when X is observed, and similarly for Y.
(b) In the case $a = b$, $\lambda = 1/2$, would the statistician prefer to observe X or Y?
(c) For general $a \neq b$, $\lambda \in (0,1)$ would the statistician prefer to observe X or Y?

5. **Optional bonus problem 1:** Let $\Theta = \mathcal{A} = \{0,1\}$, assume the (hypothesis testing) loss function $L(0,0) = L(1,1) = 0$, $L(0,1) = L(1,0) = 1$. Suppose that we observe the random variable X with the discrete distribution $P_\theta(X = x) = 2^{-(x+\theta)} 1_{\mathbb{Z}^+ \cap [1-\theta,\infty)}(x)$. (a) Describe the set of all non-randomized decision rules.
(b) Plot the risk set $\mathcal{R}$ in the plane. Which non-randomized rules are admissible? Why?
(c) Can you find a non-randomized minimax rule?
(d) What decision rules result from a Neyman-Pearson approach?
Hint: every number $z \in [0,1]$ has a diadic representation of the form $\sum_{x=1}^\infty d(x) 2^{-x}$ where $d(x) = 0$ or 1.

6. **Optional bonus problem 2:** Let X be a random variable with distribution function F having finite first moment: $E|X| < \infty$.
(a) Show that $f(b) \equiv E|X - b|$ is minimized by $b =$ any median of the distribution F of X. [A median m of F is any value satisfying $F(m) = P(X \le m) \ge 1/2$ and $1 - F(m-) = P(X \ge m) \ge 1/2$; see Lehmann and Casella, TPE, page 62, problems 1.7 and 1.8.]
(b) For $0 < \tau < 1$, let $\rho_\tau(x) = x(\tau - 1_{(-\infty,0)}(x))$. Consider minimizing

$$M_\tau(\theta) = E\rho_\tau(X - \theta)$$

with respect to θ. Show that the solution $\theta_0 = \theta_0(F)$ is given by the $\tau-$th quantile of F: $\theta_0(F) = F^{-1}(\tau)$.

7. **Optional bonus problem 3:** Suppose that $\Theta = \{\theta_1, \theta_2\}$, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, and that the loss function $L(\theta, a)$ is given by the following table:

θ/a	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

Further suppose that $P_{\theta_j}(X = 0) = 1$ for $j = 1, 2$.
(a) Find the decision risk set $\mathcal{R}$.
(b) Find the decision rules that are Bayes with respect to the prior distribution $\lambda = (1, 0)$.
(c) Show that the rule d_0 for which $R(\theta_1, d_0) = 1$ and $R(\theta_2, d_0) = 1$ is Bayes with respect to $\lambda = (1,0)$ and also minimax, but that it is not admissible.
(d) Relate this example to our theorem about admissibility of Bayes rules.