Statistics 582, Problem Set 8 Solutions

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1. A random variable X takes on the values $1,\ 2,\ 3,\ 4$ with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.18	.06	.36	.40
$p_1(x)$	.36	.18	.24	.22

(a) Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
(b) Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1-\phi)$.

Solution: (a) Now $p_1(x)/p_0(x) = 2,\ 3,\ 2/3,\ 22/40 = 11/20$, according as $x = 1,\ 2,\ 3,\ 4$, so a MP test of size $\alpha = .2$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 2 \\ .14/.18, & \text{if } x = 1 \\ 0, & \text{if } x = 3, 4. \end{cases}$$

Then

$$E_0\phi(X) = P_0(X = 2) + \frac{.14}{.18} P_0(X = 1) = .06 + \frac{.14}{.18}.18 = .2,$$

while

$$\text{Power} = E_1\phi(X) = P_1(X = 2) + \frac{.14}{.18} P_1(X = 1) = .18 + \frac{.14}{.18}.36 = .18 + .28 = .46.$$

(b) A Bayes rule (test) ϕ_λ with respect to the prior $\underline{\lambda} = (1/2, 1/2)$ is given by

$$\begin{aligned} \phi_\lambda(x) &= \begin{cases} 1, & \text{if } p_1(x) \ge p_0(x) \\ 0, & \text{if } p_1(x) < p_0(x) \end{cases} \\ &= \begin{cases} 1, & \text{if } x = 1,\ 2 \\ 0, & \text{if } x = 3, 4\ . \end{cases} \end{aligned}$$

Then $R(0, \phi_\lambda = E_0\phi_\lambda(X) = .24$, $R(1, \phi_\lambda) = E_1(1 - \phi_\lambda(X)) = .46$, and hence the Bayes risk is $\mathcal{R}(\Lambda, \phi_\lambda) = (.24 + .46)/2 = .35$ Note that the sum of the two types of error is

$$R(0, \phi_\lambda) + R(1, \phi_\lambda) = .7 = \int p_0 \wedge p_1 d\mu = 1 - d_{TV}(P_0, P_1),$$

so this result agrees with the solution of a previous problem Also note that for the Neyman Pearson test with $\alpha = .2$, the sum of the two types of error is $.2 + .54 = .74 > .7$.

2. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \; P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

Solution: Here the table of likelihood ratios is as follows:

x	1	2	3	4	5	6
$p_1(x)/p_0(x)$	2	5/2	4	2	∞	78/98
$p_2(x)/p_0(x)$	3	5/2	6	0	∞	72/98

(i) For $\alpha = .01$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1\{x = 5\} + (1/2)1\{x = 3\},$$
$$\phi_2(x) = 1\{x = 5\} + (1/2)1\{x = 3\},$$

so $\phi_1 = \phi_2$ is Uniformly most powerful.
(ii) For $\alpha = .05$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,3,5\}}(x),$$

so there is no UMP test of P_0 versus P_1 and P_2 at this level.
(iii) For $\alpha = .07$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,2,3,5\}},$$

so by taking $\gamma(x) = 1\{x = 1\}$, $\phi_1(x) = \phi_2(x)$, and this test is Uniformly Most Powerful for testing P_0 versus P_1 and P_2.

3. (Problem 3.7, Lehmann and Romano, TSH, page 94, modified) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	$\theta/2$	θ	$.9-\theta$	$.1-\theta/2$

where $0<\theta<.2$. For testing $H:\theta=.1$ against $\theta>.1$ at level $\alpha=.05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0)=1$, $\phi(1)=\phi(2)=\phi(3)=0$;
(ii) $\phi(1)=.5$, $\phi(0)=\phi(2)=\phi(3)=0$;
(iii) $\phi(3)=1$, $\phi(0)=\phi(1)=\phi(2)=0$.

Solution: The likelihood ratios $P_{\theta'}(X=x)/P_\theta(X=x)$

x	0	1	2	3
$P_\theta(X=x)$	$\theta/2$	θ	$.9-\theta$	$.1-\theta/2$
$\frac{P_{\theta'}(X=x)}{P_\theta(X=x)}$	$\frac{\theta'}{\theta}$	$\frac{\theta'}{\theta}$	$\frac{9-10\theta'}{9-10\theta}$	$\frac{1-5\theta}{1-5\theta'}$

It is easy to check that

$$\frac{\theta'}{\theta}>\frac{9-10\theta'}{9-10\theta}>\frac{1-5\theta'}{1-5\theta}$$

Hence this family has monotone decreasing likelihood ratio in x (though not strictly), and strictly decreasing likelihood ratio in

$$\begin{aligned} T(x) &= 1\{x=0\}+1\{x=1\}+2\cdot 1\{x=2\}+3\cdot 1\{x=3\} \\ &= x1\{x>0\}+1\{x=0\}. \end{aligned}$$

It follows from the Karlin - Rubin theorem that a UMP test of $H:\theta\le\theta_0=.05$ (of its level) is given by

$$\phi(X)=1_{[T(X)<k]}+\gamma(X)1_{[T(X)=k]}. \tag{1}$$

(i) Note that the test $\phi_1(X)=1\{X=0\}$ is of the form (1) with $k=1$ and $\gamma(X)=1\{X=0\}$ and it has level $\alpha=.05$; hence it is a UMP test of H versus K. The power of ϕ_1 is given by $\beta_1(\theta)\equiv E_\theta\phi_1(X)=\theta/2$.
(ii) The test $\phi_2(X)=(.5)1\{X=1\}$ is also of the form (1) with $k=1$ and $\gamma(X)=.5\cdot 1\{X=1\}$ and it has level $\alpha=.05$. Hence it is also a UMP test of H versus K. The power of ϕ_2 is given by $\beta_2(\theta)\equiv E_\theta\phi_2(X)=\theta/2$.
(iii) The test $\phi_3(X)=1\{X=3\}$ is clearly not of the form (1). It has power function $\beta_3(\theta)=E_\theta\phi_3(X)=.1-\theta/2$, so $\beta_3(.05)=.05$, but $\beta_3(\theta)>.05$ for $\theta<.10$ while $\beta_3(\theta)<.05$ for $\theta>\theta_0=.10$. In fact, this is a UMP test of $\tilde{H}:\theta\ge\theta_0$ versus $\tilde{K}:\theta<\theta_0$.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
(a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
(b) What happens to the risk body as n grows or as $\sigma_0 \to 0$?
(c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
(d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : \ n \ge 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?

Solution: (a) If $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma_0)$, to find optimal tests ϕ we can reduce (by sufficiency) to consideration of $\overline{X} \sim N(\theta, \sigma_0^2/n)$. My favorite family of tests (in fact the most powerful tests) of H versus K are the tests $\phi_c(\underline{X}) = 1\{\overline{X} > c\}$.For these tests

$$\begin{aligned} R(0, \phi_c) &= E_0\phi_c(\underline{X}) = P_0(\overline{X} > c) \\ &= P_0(\sqrt{n}(\overline{X} - 0)/\sigma_0 > \sqrt{n}c/\sigma_0) \\ &= 1 - \Phi(\sqrt{n}c/\sigma_0) \end{aligned}$$

and

$$\begin{aligned} R(1, \phi_c) &= E_1(1 - \phi_c(\underline{X})) \\ &= P_1(\overline{X} \le c) = P_1(\sqrt{n}(\overline{X} - 1) \le \sqrt{n}(c - 1)) \\ &= \Phi(\sqrt{n}(c - 1)/\sigma_0). \end{aligned}$$

Since these tests are MP for testing H versus K, there are no points with risks below the curve given by $\{(R(0, \phi_c), R(1, \phi_c)) : \ c \in \mathbb{R}\}$; this is the lower boundary of the risk body. Note that the tests $\phi_{\text{ignore}}(\underline{X}) \equiv \alpha$ have risks $R(0, \phi_{\text{ignore}}) = \alpha$, $R(1, \phi_{\text{ignore}}) = 1 - \alpha$. Thus the line $\{(\alpha, 1 - \alpha) : \ \alpha \in [0, 1]\}$ is in the risk body. Furthermore, note that the tests $\phi_c'(\underline{X}) \equiv 1 - \phi_c(\underline{X}) = 1\{\overline{X} \le c\}$ are MP for testing $H : \theta = 0$ versus $K' : \theta = \theta_1 < 0$, and by the Karlin - Rubin theorem these tests minimize the power function at points $\theta = \theta_1$ in the class of all tests with fixed power function (say at α) at $\theta = \theta_0$. Since

$$\text{Power}_{\phi_c'}(\theta) = E_\theta \phi_c' = 1 - R(\theta, \phi_c),$$

this says that the tests ϕ_c' maximize $R(1, \phi_c)$ over tests ϕ with $R(0, \phi) = \alpha$. Hence there are no points in the risk body with risks above the curve given by $\{(1 - R(0, \phi_c), 1 - R(1, \phi_c)) : \ c \in \mathbb{R}\}$.
(b) As n grows or $\sigma_0 \to 0$ the risk body expands out toward the boundary of the square $[0, 1]^2$; see the plots below.
(c) As $\theta_1 \to \theta_0 = 0$, the risk body contracts toward the diagonal line $(\alpha, 1 - \alpha)$ – since the testing problem becomes harder. See the plots below.

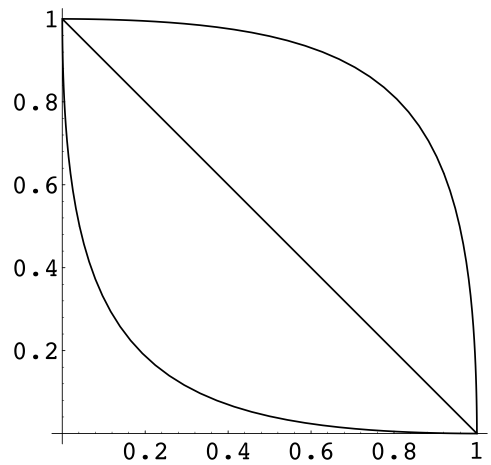

Figure 1: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = 1$.

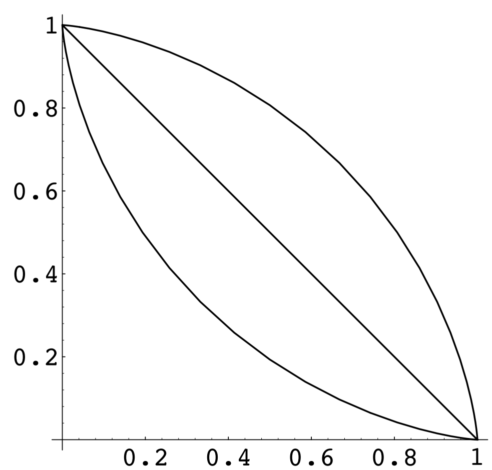

Figure 2: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = .5$.

5. Consider the logistic distributions with location parameter θ having density

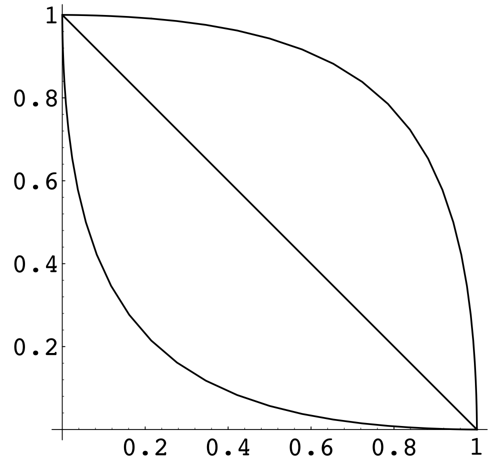

Figure 3: Risks for normal mean test, $n = 10$, $\sigma_0 = 1$, $\theta_1 = .5$.

$p_\theta(x) = g(x - \theta)$ where

$$g(x) = \frac{\exp(x)}{(1 + \exp(x))^2} = \frac{1}{2(1 + \cosh(x))} = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

(a) Show that the family $\{p_\theta : \theta \in \mathbb{R}\}$ has monotone likelihood ratio in x.
(b) Unfortunately the result of (a) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ as in (a), then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

Solution: (a) Let $\theta' > \theta$. Then the ratio of densities is given by

$$\begin{aligned} \frac{p_{\theta'}(x)}{p_\theta(x)} &= \frac{e^{x-\theta'}}{(1 + e^{x-\theta'})^2} \cdot \frac{(1 + e^{x-\theta})^2}{e^{x-\theta}} \\ &= e^{\theta-\theta'} \left(\frac{1 + e^{x-\theta}}{1 + e^{x-\theta'}} \right)^2 . \end{aligned}$$

This is a monotone increasing function of x if and only if its logarithm is a monotone increasing function of x. The logarithm is given by

$$\begin{aligned} \log \left(\frac{p_{\theta'}(x)}{p_\theta(x)} \right) &\equiv R(x; \theta, \theta') \equiv R(x) \\ &= \theta - \theta' + 2 \log \left(\frac{1 + e^{x-\theta}}{1 + e^{x-\theta'}} \right) \\ &= \theta - \theta' + 2 \left\{ \log(1 + e^{x-\theta}) - \log(1 + e^{x-\theta'}) \right\} , \end{aligned}$$

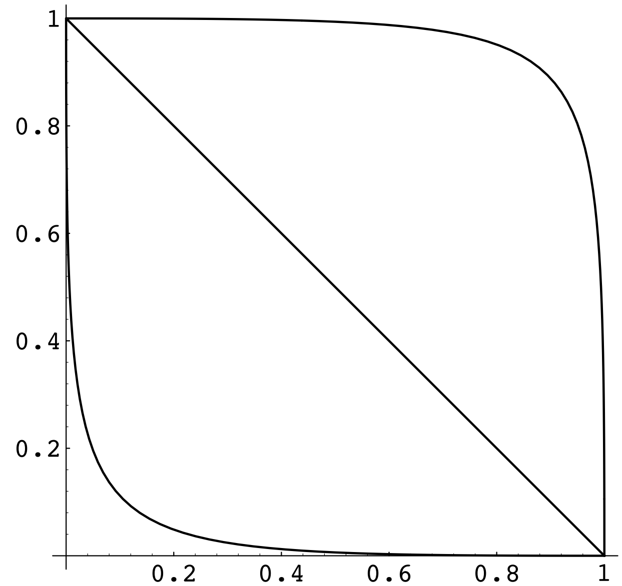

Figure 4: Risks for normal mean test, $n = 25$, $\sigma_0 = 1$, $\theta_1 = .5$.

where R has derivative (with respect to x) R' given by

$$\begin{aligned} R'(x) &= 2\left\{ \frac{e^{x-\theta}}{1+e^{x-\theta}} - \frac{e^{x-\theta'}}{1+e^{x-\theta'}} \right\} \\ &= \frac{2e^x}{(1+e^{x-\theta})(1+e^{x-\theta'})} \cdot (e^{-\theta} - e^{-\theta'}) \\ &> 0. \end{aligned}$$

Thus the family $\{p_\theta\}$ has monotone likelihood ratio in $T(x) = x$.
Alternatively, this follows from the fact that g is log-concave and our (new) Example 6.1.3: $h(x) = \log g(x) = x - 2\log(1+e^x)$ has

$$\begin{aligned} & h'(x) = 1 - \frac{2e^x}{1+e^x} = 1 - 2\frac{1}{1+e^{-x}} = 1 - 2G(x), \\ & h''(x) = -2g(x) < 0 \quad \text{for all} \quad x. \end{aligned}$$

(b) As in Example 6.1.5, the locally best test is the one-sided score test, reject if $S_n(\theta_0) > k$ where

$$S_n(\theta_0) \;=\; \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\mathbf{l}}_\theta(X_i;\theta_0).$$

Straightforward calculation yields

$$\dot{\mathbf{l}}_\theta(x) = 2\left\{ \frac{1}{1+e^{-(x-\theta)}} - 1 \right\} = 2\{G(x-\theta) - 1\}$$

where $G(x) = 1/(1 + e^{-x})$ is the standard logistic distribution function. Thus for $\theta_0 = 0$ the test statistic is

$$S_n(0) = n^{-1/2} \sum_{i=1}^{n} 2 \left\{ \frac{1}{1 + e^{-X_i}} - 1 \right\},$$

and we reject for large values of $S_n(0)$. Since $S_n(0) \to_d N(0, 1/3)$ under $\theta_0 = 0$, taking the constant k to be $3^{-1/2} z_\alpha$ leads to approximate size α for large n.