Statistics 582, Final Exam Solutions

Wellner; 3/15/2011

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A level α *permutation test* (in the setting of $m + n = N$ observations $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ i.i.d. some $F \in \mathcal{F}_c$.

   (b) A *minimax decision rule* in a general decision problem with loss function $L(\theta, d)$.

   (c) A *uniformly most powerful size* α *test* of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A *uniformly most powerful unbiased size* α *test* of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (e) An *inadmissible decision rule* d.

   Solution: See course notes.

2. (30 points) **State** *any three* of the following results:

   (a) A result concerning some optimality property of $\overline{X}_n$ in the context of estimating the mean μ when $X_1, \ldots, X_n$ are i.i.d. $N_d(\mu, \sigma^2 I)$ with σ^2 known.

   (b) A conditional limit theorem about the large sample behavior of posterior distributions.

   (c) The generalized Neyman-Pearson lemma (short form).

   (d) Wald's theorem on strong consistency of maximum likelihood estimators.

   (e) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   Solution: See course notes.

3. (40 points) Let $\mathcal{P} = \{p_\theta : \ \theta \in \Theta\}$ where p_θ is a family of densities with respect to a fixed dominating measure μ defined on a sample space $\mathcal{X}$.
(a) If $\Theta \subset \mathbb{R}$ and $\mathcal{X} \subset \mathbb{R}$, define what is meant by "the family $\mathcal{P}$ has Monotone Likelihood Ratio (MLR) in x".
(b) Suppose that the densities $p_\theta(x) \equiv p(x, \theta)$ have a second mixed partial derivative and that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial \theta} \log p(x, \theta) \geq 0$$

for all $x \in \mathbb{R}$ and $\theta \in \Theta$. Show that the inequality in the last display implies that $\mathcal{P}$ has monotone likelihood ratio. [Hint: use the fundamental theorem of calculus twice.]

Solution: (a) $\mathcal{P}$ has monotone likelihood ratio in x if for each $\theta_1 < \theta_2$, $\theta 1, \theta_2 \in \Theta$, the ratio $p_{\theta_2}(x)/p_{\theta_1}(x)$ is an increasing function of x.
(b) Let $\theta_1 < \theta_2$ and $x_1 < x_2$. Then integrating with respect to x across the given inequality (from x_1 to x_2) yields

$$\begin{aligned} 0 \ &\leq \ \int_{x_1}^{x_2} \frac{\partial}{\partial x}\frac{\partial}{\partial \theta} \log p(x, \theta) \ dx \\ &= \ \frac{\partial}{\partial \theta} \log p(x_2, \theta) - \frac{\partial}{\partial \theta} \log p(x_1, \theta) \\ &= \ \frac{\partial}{\partial \theta} \log \frac{p(x_2, \theta)}{p(x_1, \theta)} \quad \text{for all} \ \ \theta. \end{aligned}$$

Integrating this last inequality with respect to θ (from θ_1 to θ_2) yields

$$\begin{aligned} 0 \ &\leq \ \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \log \frac{p(x_2, \theta)}{p(x_1, \theta)} d\theta \\ &= \ \log \frac{p(x_2, \theta_2)}{p(x_1, \theta_2)} - \log \frac{p(x_2, \theta_1)}{p(x_1, \theta_1)} \\ &= \ \log \frac{p(x_2, \theta_2)/p(x_2, \theta_1)}{p(x_1, \theta_2)/p(x_1, \theta_1)}. \end{aligned}$$

Thus it follows by exponentiating both sides of the last display that

$$1 \leq \frac{p(x_2, \theta_2)/p(x_2, \theta_1)}{p(x_1, \theta_2)/p(x_1, \theta_1)},$$

and hence that $p(x_1, \theta_2)/p(x_1, \theta_1) \leq p(x_2, \theta_2)/p(x_2, \theta_1)$. We conclude that $\mathcal{P}$ has MLR if the stated condition on the second mixed partial derivative holds.

Do either problem 4 or problem 5.

4. (40 points)
(a) Suppose that $X \sim p(\cdot, \theta)$ where $p(\cdot, \theta)$ is given by

$$p(x, \theta) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} \cdot \frac{\theta}{1-(1-\theta^2)x} 1_{(0,1)}(x), \quad \theta \in (0, \infty). \tag{1}$$

(This is the distribution of the total time spent in $(0, \infty)$ up to time $t = 1$ by a skew Brownian motion process with skewing parameter $\theta = \sigma_+/\sigma_-$ where σ_+^2 is the variance parameter for the positive space axis and σ_-^2 is the variance parameter for the negative space axis.) Find the density $q(t, \theta)$ of $T = 1/X$.
(b) Use the sufficient condition given in problem 3 to show that $q(t, \theta)$ has MLR in t.
(c) Find a UMP size $\alpha = .05$ test of $H : \theta \leq 1 \equiv \theta_0$ versus $K : \theta > 1$. Specify your test completely, including the constants.
(d) Compute the power function of the UMP test you derived in (c) as explicitly as possible.

Solution: (a) Now $P_\theta(T \leq t) = P_\theta(1/X \leq t) = P_\theta(X \geq 1/t)$, and hence the density $q(t, \theta)$ of T is given by

$$\begin{aligned}
q(t, \theta) &= p(1/t, \theta)t^{-2} = \frac{1}{\pi} \frac{1}{\sqrt{(1/t)(1-(1/t))}} \cdot \frac{\theta}{1-(1-\theta^2)(1/t)} \cdot t^{-2} 1_{(1,\infty)}(t) \\
&= \frac{1}{\pi} \frac{\theta}{\sqrt{(t-1)}} \cdot \frac{1}{t-(1-\theta^2)} 1_{(1,\infty)}(t).
\end{aligned}$$

(b) Now

$$\log q(t, \theta) = \log(1/\pi) - (1/2)\log(t(t-1)) - \log(t-(1-\theta^2)),$$

so

$$\begin{aligned}
\frac{\partial}{\partial\theta} \log q(t, \theta) &= -\frac{2\theta}{t-(1-\theta^2))}, \quad \text{and} \\
\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} \log q(t, \theta) &= \frac{2\theta}{(t-(1-\theta^2))^2} \geq 0
\end{aligned}$$

so the condition of problem 3 holds and $q(t, \theta)$ has MLR in t.
(c) Now the UMP test H versus K of size $\alpha = .05$ rejects H if $T > c_\alpha$ where c_α is determined by

$$\alpha = P_1(T > c_\alpha) = P_1(X < 1/c_\alpha) = \int_0^{1/c_\alpha} \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} dx$$

But since we know that $P_1(X \leq x) = (2/\pi)\arcsin(\sqrt{x})$, this yields $1/\tilde{c} = 1/\tilde{c}_\alpha = (\sin(\pi\alpha/2))^2 = (\sin(\pi/40))^2$.
(d) The power function of the test in (b) is just

$$\begin{aligned}
\beta_\phi(\theta) &= E_\theta\phi(X) = P_\theta(X < 1/\tilde{c}_\alpha) = \int_0^{1/\tilde{c}_\alpha} p(x;\theta)dx \\
&= \int_0^{1/\tilde{c}_\alpha} \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}} \cdot \frac{\theta}{1-(1-\theta^2)x}dx \\
&= 1 - \frac{2}{\pi}\arctan\left(\frac{\cot(\alpha\pi/2)}{\theta}\right)
\end{aligned}$$

after three changes of variables: $x = t^2$; followed by $t = 1/y$; followed by $v = \sqrt{y^2-1}$. This last formula is due to Anna Klimova, who computed it during the exam in (2009). See Figure **??** for a plot of this power function.

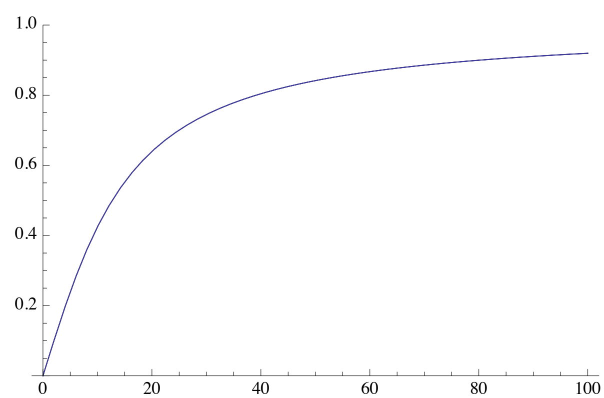

Figure 1: Plot of power, UMP test

5. (40 points)
(a) Suppose that T is a random variable with density $p(\cdot;\theta)$ given by

$$p(t;\theta) = \frac{1}{\sqrt{2\pi}}\frac{\theta^{1/2}}{t^{3/2}}\exp\left(-\frac{\theta}{2t}\right)1_{(0,\infty)}(t), \quad \theta > 0. \tag{2}$$

This is the time required for standard Brownian motion $\mathbb{S}$ to hit the level $\sqrt{\theta}$ starting from 0. Note that

$$\begin{aligned}
P_\theta(T \leq t) &= 2P(\mathbb{S}(t) \geq \sqrt{\theta}) = \int_{\sqrt{\theta}}^\infty \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)dy \\
&= 2P(N(0,1) \geq \sqrt{\theta/t}).
\end{aligned} \tag{3}$$

Use the sufficient condition given in problem 3 to show that $p(t;\theta)$ has MLR in t.
(b) Find the UMP size .01 test of $H : \theta \le 1 \equiv \theta_0$ versus $K : \theta > 1$. Specify your test completely, including the constants.
(c) Compute the power function of the UMP test in (b) as explicitly as possible.

Solution: (a) Now

$$\log p(t;\theta) = -(1/2)\log(2\pi) + \frac{1}{2}\log\theta - \frac{3}{2}\log t - \frac{\theta}{2t},$$

so

$$\begin{aligned}\frac{\partial}{\partial\theta}\log p(t;\theta) &= \frac{1}{2\theta} - \frac{1}{2t}, \quad \text{and}\\ \frac{\partial}{\partial t}\frac{\partial}{\partial\theta}\log p(t;\theta) &= \frac{1}{2t^2} \ge 0.\end{aligned}$$

Thus the sufficient condition of problem 3 implies that $\{p(\cdot,\theta)\}$ has MLR.
(b) By the Karlin-Rubin theorem, the UMP test of H versus K of size $\alpha = .01$ is of the form "reject H if $T > c_\alpha$ where c_α satisfies

$$\begin{aligned}\alpha &= P_1(T > c_\alpha) = 1 - P_1(T \le c_\alpha) = 1 - 2P(\mathbb{S}(c_\alpha) \ge 1)\\ &= 1 - 2P(\sqrt{c_\alpha}Z \ge 1) = 1 - 2P(Z \ge 1/\sqrt{c_\alpha});\end{aligned}$$

i.e.

$$\frac{1-\alpha}{2} = P(Z \ge 1/\sqrt{c_\alpha}) = 1 - \Phi(1/\sqrt{c_\alpha}),$$

or $\Phi^{-1}(1/\sqrt{c_\alpha}) = 1 - (1-\alpha)/2 = (1+\alpha)/2$, and hence

$$\frac{1}{\sqrt{c_\alpha}} = \Phi^{-1}\left(\frac{1+\alpha}{2}\right), \quad \text{or} \quad c_\alpha = \frac{1}{\Phi^{-1}\left(\frac{1+\alpha}{2}\right)^2}.$$

When $\alpha = .01$, this yields $c_{.01} = 6365.86$
(c) The power function of this test is given by

$$\begin{aligned}\beta_\phi(\theta) &= P_\theta(T > c_\alpha) = \int_{c_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}}\frac{\theta^{1/2}}{t^{3/2}}\exp\left(-\frac{\theta}{2t}\right)dt\\ &= 1 - P_\theta(T \le c_\alpha)\\ &= 1 - 2P(Z \ge \sqrt{\theta/c_\alpha}) \ \text{ by } \ (\textbf{??})\\ &= 1 - 2(1 - \Phi(\sqrt{\theta/c_\alpha})) = 1 - 2(1 - \Phi(\sqrt{\theta}\Phi^{-1}((1+\alpha)/2)))\\ &= 2\Phi(\sqrt{\theta}\Phi^{-1}((1+\alpha)/2)) - 1.\end{aligned}$$

Note that the right side converges to 1 as $\theta \to \infty$ and that it equals $2\Phi(\Phi^{-1}((1+\alpha)/2)) - 1 = 2(1+\alpha)/2 - 1 = \alpha$ when $\theta = 1$.

Do either problems 6 or problem 7.

6. (40 points) Suppose that X has distribution function F and Y has distribution function G on $[0, \infty)$. where X and Y are independent. Suppose that we observe $Z = X \wedge Y = \min\{X, Y\}$ and $\Delta = 1\{X \le Y\}$. Suppose that X has distribution function F on $[0, \infty)$.
(a) Express the sub-distribution functions $F^{uc}(z, 1) = P(Z \le z, \Delta = 1)$ and $F^{c}(z, 0) = P(Z \le z, \Delta = 0)$ in terms of F and G.
(b) Express the survival function $1 - H(z) = P(Z > z)$ in terms of F and G.
(c) Express the cumulative hazard function $\Lambda = \Lambda_F$ of X in terms of the sub-distribution functions F^{uc}, F^c, and the marginal survival function $1 - H$ of Z.
(d) Based on the identity you found in (c), suggest a nonparametric maximum likelihood estimator of Λ_F.

Solution: (a) Now

$$\begin{aligned} F^{uc}(z) &= P(Z \le z, \Delta = 1) = P(X \le z, X \le Y) = E\{1_{[X \le z]} 1_{[Y \ge X]}\} \\ &= E\{E\{1_{[X \le z]} 1_{[Y \ge X]} | X\}\} \\ &= E\{1_{[X \le z]} E\{1_{[Y \ge X]} | X\}\} = E\{1_{[X \le z]}(1 - G(X-))\} \\ &= \int_{[0,z]} (1 - G(x-)) dF(x). \end{aligned}$$

Similarly,

$$\begin{aligned} F^{c}(z) &= P(Z \le z, \Delta = 0) = P(Y \le x, X > Y) = E\{1_{[Y \le z]} 1_{[Y < X]}\} \\ &= E\{E\{1_{[Y \le z]} 1_{[Y < X]} | Y\}\} = E\{1_{[Y \le z]} E\{1_{[Y < X]} | Y\}\} \\ &= E\{1_{[Y \le z]}(1 - F(Y))\} = \int_{[0,z]} (1 - F(y)) dG(y). \end{aligned}$$

(b) Also, $1 - H(z) = P(Z > z) = P(X > z, Y > z) = P(X > z)P(Y > z) = (1 - F(z))(1 - G(z))$.
(c) The cumulative hazard function $\Lambda = \Lambda_F$ is given by

$$\begin{aligned} \Lambda_F(x) &= \int_{[0,x]} \frac{1}{1 - F(t-)} dF(t) = \int_{[0,x]} \frac{1 - G(t-)}{(1 - F(t-))(1 - G(t-))} dF(t) \\ &= \int_{[0,x]} \frac{1}{1 - H(t-)} dF^{uc}(t) \end{aligned}$$

by using (a) and (b) in the last step.
(d) If $(Z_1, \Delta_1), \ldots, (Z_n, \Delta_n)$ are i.i.d. as (Z, Δ), then the nonparametric MLE's

of F^{uc}, F^c and H are given by

$$\mathbb{F}_n^{uc}(z) = \frac{1}{n}\sum_{i=1}^{n} \Delta_i 1\{Z_i \le z\},$$
$$\mathbb{F}_n^{c}(z) = \frac{1}{n}\sum_{i=1}^{n} (1-\Delta_i) 1\{Z_i \le z\}, \quad \text{and}$$
$$\mathbb{H}_n(z) = \frac{1}{n}\sum_{i=1}^{n} 1\{Z_i \le z\}.$$

Then the nonparametric maximum likelihood estimator of Λ_F is given by

$$\widehat{\Lambda}_n(x) = \int_{[0,x]} \frac{1}{1-\mathbb{H}_n(t-)} d\mathbb{F}_n^{uc}(t).$$

This is the Nelson-Aalen estimator of Λ_F.

7. (40 points) Consider testing

$$H_c: \ X_1, \ldots, X_m, Y_1, \ldots, Y_n \ \text{ are i.i.d. } \ F \in \mathcal{F}_c$$

versus

$$K_1: \ X_1, \ldots, X_m, Y_1, \ldots, Y_n \ \text{ have joint density } \ h$$

where h is given by

$$\begin{aligned} h(\underline{x}, \underline{y}, \mu, \nu) \ &= \ \mu^{-m} \exp\left(-\sum_1^m x_i/\mu\right) \prod_{i=1}^{m} 1_{(0,\infty)}(x_i) \\ &\quad \cdot \nu^{-n} \exp\left(-\sum_1^n y_j/\nu\right) \prod_{j=1}^{n} 1_{(0,\infty)}(y_j) \end{aligned}$$

where $\nu > \mu$ are both fixed.
(a) Find a MP similar test of H_c versus K_1; express your test in terms of either $\overline{Y}_n - \overline{X}_m$ or $\sum_{j=1}^n Y_j$.
(b) Does the resulting test in (a) depend on the particular $\nu > \mu$ which were specified? State the resulting property of the test derived in (a).
(c) Use the Wald-Wolfowitz-Noether-Hajék CLT to approximate the critical points of the test you derived in (a) and (b) under some appropriate conditions.

Solution:
(a) To find a MP similar test of H_c we condition on the order statistics $\underline{Z}$ of the pooled sample, $\underline{Z} = (Z_1, \ldots, Z_N)$ with $N = m + n$. To find a most powerful similar test of H_c versus K_1, we reject H_c for those permutations $\underline{z}'$ of $\underline{Z}$ for which

$$\mu^{-m}\nu^{-n}\exp\left(-\frac{1}{\mu}\sum x_i - \frac{1}{\nu}\sum y_j\right)$$

is large; or, equivalently, if

$$\frac{1}{\mu}\sum x_i + \frac{1}{\nu}\sum y_j$$

is small; or equivalently if

$$\left(\frac{1}{\nu} - \frac{1}{\mu}\right)\sum y_j + \frac{1}{\mu}\left(\sum x_i + \sum y_j\right)$$

is small; or, equivalently (since $\nu > \mu$ for the alternative K_1), if

$$\sum_{j=1}^{n} y_j$$

is large. Thus we reject if $\sum_{j=1}^{n} Y_j > c_\alpha(\underline{Z})$ where $c_\alpha(\underline{Z})$ is chosen so that we reject for exactly $I = \alpha N!$ of the $N!$ permutations of $\underline{Z}$.
(b) The resulting test does not depend on which particular $\nu > \mu$ are used to specify K_1, so in fact the resulting test is UMP similar for testing H_c versus $K \equiv \cup_{\mu<\nu} K_{1,\mu,\nu}$.
(c) Assuming that $E_F X_1^2 < \infty$ and $E_G Y_1^2 < \infty$ we know that the order statistics $Z_1, \ldots, Z_N$ of the X's and Y's satisfy the Noether condition

$$\frac{\max_{1\le i\le N}|Z_i - \overline{Z}_N|^2}{\sum_{j=1}^{N}|Z_n - \overline{Z}_N|^2} \rightarrow_{a.s.} 0.$$

Thus if $0 < \underline{\lim}(m/N) \le \overline{lim}(m/N) < 1$, the WWNH finite-sampling CLT implies that

$$\frac{\overline{Y}_n - \overline{z}_N}{\sigma_N} \rightarrow_d N(0,1)$$

where $\sigma_N^2 = (1-(n-1)/(N-1))\sigma_z^2/n$. Hence we can approximate the upper α critical point $c_\alpha(\underline{z})$ for the permutation test based on $\overline{Y}_n$ by $\sigma_N z_\alpha + \overline{z}_N$.

Do **either** problem 8 **or** problem 9.

8. (40 points) Suppose that $X_i \sim N(\Delta i, 1)$, $i = 1, \ldots, n$ with $\Delta > 0$.
(a) Find a UMP size α test of $H : \Delta \le 0$ versus $K : \ \Delta > 0$.
(b) Compute the power function of the test you found in (a) as explicitly as possible.
(c) Find an alternative test of size α of H versus K based on $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$.
(d) Compute the power function of the test you found in (c) as explicitly as possible.
(e) Compare the power functions you found in (b) and (d). Which test do you prefer?

Solution: (a) The joint density of the X_i's is given by

$$\begin{aligned} p_\Delta(\underline{x}) &= (2\pi)^{-n/2} \exp\left(- \sum_1^n (x_i - \Delta i)^2/2 \right) \\ &= (2\pi)^{-n/2} \exp\left(\Delta \sum_1^n i x_i - \frac{1}{2}\Delta^2 \sum_1^n i^2 \right) \exp(-\sum_1^n x_i^2/2) \end{aligned}$$

and this has MLR in $T(\underline{X}) = \sum_1^n iX_i$. Thus by the Karlin - Rubin theorem, the UMP test of H versus K is of the form $\phi(\underline{X}) = 1\{T(\underline{X}) > k\}$ where k is chosen so that $E_0\phi(\underline{X}) = P_0(T(\underline{X}) > k) = \alpha$. Now $T \sim N(ET, Var(T)) = N(\Delta \sum_1^n i^2 \, , \, \sum_1^n i^2)$ where

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \equiv \sigma_n^2.$$

Thus $P_0(T > k) = P_0(N(0, \sigma_n^2) > k) = \alpha$ if $k = z_\alpha \sigma_n$.
(b) The power function of the UMP test derived in (a) is

$$\begin{aligned} \beta_\phi(\Delta) &= P_\Delta(T > z_\alpha \sigma_n) \\ &= P_\Delta\left(\frac{T - \Delta\sigma_n^2}{\sigma_n} > z_\alpha - \Delta\sigma_n \right) \\ &= P(Z > z_\alpha - \Delta\sigma_n). \end{aligned}$$

(c) For the alternative test based on $\overline{X}_n$ we have

$$E_\Delta(\overline{X}_n) = n^{-1} \sum_{i=1}^n i\Delta = (n+1)\Delta/2,$$

$$Var_\Delta(\overline{X}_n) = n^{-2} \sum_{i=1}^n 1 = n^{-1},$$

so we can use the test $\phi^*(\underline{X}) = 1\{\overline{X}_n > k^*_\alpha\}$ where k^*_α is determined by

$$\beta_{\phi^*}(0) = P_0(\overline{X}_n > k^*_\alpha) = P_0\left(\frac{\overline{X}_n - 0\cdot(n+1)/2}{\sqrt{1/n}} > \frac{k^*_\alpha}{\sqrt{1/n}}\right) = \alpha$$

and hence $k^*_\alpha = z_\alpha n^{-1/2}$.
(d) The power function ϕ^* of the test based on $\overline{X}_n$ is given by

$$\begin{aligned}
\beta_{\phi^*}(\Delta) &= P_\Delta(\overline{X}_n > z_\alpha n^{-1/2}) \\
&= P_\Delta\left(\frac{\overline{X}_n - \Delta(n+1)/2}{\sqrt{1/n}} > \frac{z_\alpha n^{-1/2} - \Delta(n+1)/2}{1/\sqrt{n}}\right) \\
&= P\left(Z > z_\alpha - \sqrt{n}(n+2)\Delta\right).
\end{aligned}$$

(e) Note that $\sqrt{n(n+1)(2n+1)/6} > \sqrt{n}(n+1)/2$ since $2n+1 > (3/2)(n+1)$ for $n > 1$ Hence we prefer the UMP test derived in (a) and (b) for $n > 1$. (When $n = 1$, the two tests coincide.)

9. (40 points) Suppose that $X_1, \ldots, X_m$ are i.i.d. Exponential$(1/\mu)$ (with density $p(x;\mu) = \mu^{-1}\exp(-x/\mu)1_{(0,\infty)}(x)$) and suppose that $Y_1, \ldots, Y_n$ are i.i.d. Exponential$(1/\nu)$ (with density $p(y;\nu) = \nu^{-1}\exp(-y/\nu)1_{(0,\infty)}(y)$) and independent of the X_i's. Consider testing $H:\ \nu \le \mu$ versus $K:\ \nu > \mu$.
(a) Find a sufficient statistic $T = T(\underline{X}, \underline{Y})$ for $\theta \in \Theta_B = \overline{\Theta}_0 \cap \overline{\Theta}_1$.
(b) Find a UMP unbiased test of size $\alpha \in (0,1)$ of H versus K in the form of a conditional test.
(c) Show that the UMPU test you found in (b) can be carried out unconditionally by use of an appropriate ancillary statistic and Basu's theorem.
(d) Compute the power function of your test as explicitly as possible.

Solution:
(a) The joint density of the data is given by

$$\begin{aligned}
p_{\mu,\nu}(\underline{x}, \underline{y}) &= \mu^{-m}\nu^{-n}\exp\left(-\frac{1}{\mu}\sum_{i=1}^m x_i - \frac{1}{\nu}\sum_{j=1}^n y_j\right) \\
&= \mu^{-m}\nu^{-n}\exp\left(\left(\frac{1}{\mu} - \frac{1}{\nu}\right)\sum_{i=1}^n y_j - \frac{1}{\mu}\left(\sum_{i=1}^n x_i + \sum_{j=1}^n y_j\right)\right) \\
&= \mu^{-m}\nu^{-n}\exp\left(\theta U(\underline{x}, \underline{y}) + \xi T(\underline{x}, \underline{y})\right)
\end{aligned}$$

where

$$\theta \equiv \frac{1}{\mu} - \frac{1}{\nu}, \qquad U(\underline{x}, \underline{y}) = \sum_{j=1}^{n} y_j,$$
$$\xi \equiv -\frac{1}{\mu}, \qquad T(\underline{x}, \underline{y}) = \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j.$$

It follows that on the boundary set $\Theta_B = \{(\mu, \mu) : \mu > 0\}$ the statistic $T = T(\underline{X}, \underline{Y})$ is sufficient for $\xi = -1/\mu$.

(b) From the derivation in (a) we see that testing H versus K is equivalent to testing $\theta \le 0$ versus $\theta > 0$. Thus a UMP unbiased test of H versus K is given by

$$\phi(\underline{X}, \underline{Y}) = \begin{cases} 1 & \text{if } \sum_1^n Y_j > c(T) \\ \gamma(T) & \text{if } \sum_{i=1}^n \sum_1^n Y_j = c(T) \\ 0 & \text{if } \sum_{j=1}^n Y_j < c(T). \end{cases}$$

where $c(T)$, $\gamma(T)$ are chosen so that $E_{(\mu,\mu)}\phi(\underline{X}, \underline{Y}) = \alpha$.

(c) Now on the boundary $\nu = \mu$, $V = U/T = \sum Y_j / (\sum X_i + \sum Y_j) \sim \text{Beta}(n, m)$ is ancillary, and hence independent of T by Basu's theorem. Since V is a monotone function of U for each fixed $T = t$, it follows that the test in (b) can be re-expressed as $\phi(\underline{X}, \underline{Y}) = 1\{V > c\}$ where c is chosen so that $P_{\mu,\mu}(V > c) = \int_c^1 \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} v^{n-1}(1-v)^{m-1} dv = \alpha$. Alternatively, since $U \equiv \sum Y_j \sim \text{Gamma}(n, \mu)$ and $W \equiv \sum X_i \sim \text{Gamma}(m, \mu)$ on the boundary $\nu = \mu$, it follows that $2\mu U \sim \chi^2_{2n}$ and $2\mu W \sim \chi^2_{2m}$ are independent, so

$$\frac{U/(2n)}{W/(2m)} = \frac{m}{n}\frac{U}{W} \sim F_{2n,2m}$$

and, moreover

$$V = U/T = U/(U+W) = \frac{U/W}{1 + U/W}$$

is a monotone increasing function of U/W, so the UMPU test can be carried out unconditionally using tables of the $F_{2n,2m}$ distributions.

(d) Note that the power of the test derived in (b) and (c) is most easily calculated from the $F_{2n,2m}$-form explained in (c): under K we have $2\mu U \sim \chi^2_{2n}$ and $2\nu W \sim \chi^2_{2m}$ where these random variables are independent. Hence

$$\begin{aligned} \beta_\phi(\mu, \nu) &= P\left(\frac{U/(2n)}{W/(2m)} > F_{2n,2m,\alpha} \right) \\ &= P\left(\frac{2\nu}{2\mu} \cdot \frac{2\mu U/(2n)}{2\nu W/(2m)} > F_{2n,2m,\alpha} \right) \\ &= P\left(\frac{\chi^2_{2n}/(2n)}{\chi^2_{2m}/(2m)} > \frac{\mu}{\nu} F_{2n,2m,\alpha} \right). \end{aligned}$$

Note that the right side in the last display equals α when $\mu = \nu$, while it converges to 1 as $\mu/\nu \to 0$ and to 0 as $\mu/\nu \to \infty$.