Statistics 582, Problem Set 7

Wellner; 2/16/2011

Reading: Chapter 5, sections 5.7-5.8;
start reading Chapter 6 (to be handed out Friday 2/18)
Due: Wednesday, February 23, 2011.

1. Suppose that $X \sim P_\theta$ for $\theta \in \Theta \subset R^k$ has well-defined Fisher information matrix $I(\theta)$ for θ. The *Jeffreys prior* distribution Λ_J has density $\lambda_J(\theta) = \det(I(\theta))^{1/2}$ with respect to Lebesgue measure on Θ. Note that Λ_J may not be a finite measure, and even if Λ_J is a finite measure, it may not have total mass 1. If a prior distribution is a finite measure, then call it a *proper prior distribution*, and correspondingly if it is not a finite measure, call it an *improper prior distribution*. If the resulting posterior distribution is a finite measure, call it a *proper posterior distribution*, and (by convention) normalize it to have total mass 1. See Lehmann and Casella, TPE, pages 230, 234, 287, 305.
(a) Suppose that $X \sim$ Bernoulli(θ). Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.
(b) Suppose that $X \sim$ Poisson(θ) with $\theta \in (0,\infty)$. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. Is it ever a proper posterior distribution?
(c) Suppose that $X \sim$ Geometric(θ), i.e. the number of trials until the first success in i.i.d. Bernoulli trials with probability θ of success for each trial – recall Chapter 1, section 1. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. If we observe $X_1,\ldots,X_n$ i.i.d. Geometric(θ), so that $\sum X_i \sim$ Negative Binomial(n,θ) is the posterior distribution "proper" for some n?
(d) Suppose that $X \sim$ Weibull(θ) with $\theta = (\alpha,\beta) \in (0,\infty)\times(0,\infty)$ as in chapters 3 and 4. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty)^2)$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.

2. Continuation of problem 3, problem set 5: Suppose that $X_1,\ldots,X_n$ are i.i.d. Exponential(θ) (so the X's have distribution P_θ and density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on $\mathbb{R}$, and that $\theta \sim \Gamma(\alpha,\beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

In problem set 5 we found the Bayes rules with respect to squared error loss $L(\theta, a) = (\theta - a)^2$ and weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$.

(a) Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
(b) What does theorem 5.8.2 say about the limiting distribution of the Bayes rule for squared error loss (assuming that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \equiv P$ with $\theta_0 \in (0, \infty)$)?

3. Specialize the decision rule in Theorem 5.2 of the course notes to the case when P_i is the normal distribution $N_d(\mu_i, I)$, $i = 1, \ldots, k$ where $\mu_1, \ldots, \mu_k$ are distinct vectors in $\mathbb{R}^d$, $\mu_i \neq \mu_j$ for $i \neq j$. What happens if we replace I by Σ?

4. Lehmann and Casella, TPE, Problem 5.17, page 293. (Also note Problems 5.18, 5.19, 5.20, page 293.)

5. **Optional bonus problem 1:** (Birgé). Let $X = (X_0, X_1, \ldots, X_k)$ be a $(k+1)-$dimensional vector, and assume that $X \sim N_{k+1}(\theta, I_{k+1})$ where I_{k+1} denotes the $(k+1) \times (k+1)$ identity matrix. For any vector $\theta \in \mathbb{R}^{k+1}$, let θ' denote the projection of θ onto the $k-$dimensional linear space spanned by the $k-$last coordinates. Consider the subset Θ_0 of $\Theta = \mathbb{R}^{k+1}$ given by

$$\Theta_0 = \{\theta \in \mathbb{R}^{k+1} : |\theta_0| \leq k^{1/4} \ \text{ and } \ \|\theta'\| \leq 2(1 - k^{-1/4}|\theta_0|)\}.$$

(a) Show that the MLE of θ over Θ_0 is given by $\widehat{\theta_0} = 0$ and $\widehat{\theta'} = 2X'/\|X\|$ *on the event*

$$\Omega_0 \equiv \{\|X'\|^2 > 3k/4 \ \text{ and } \ |X_0| < c + 1.21\}.$$

(b) Show that $P_\theta(\Omega_0) \geq 3/4$ for all $\theta \in \Theta_0$.
(c) Let $\widetilde{\theta} = (X_0, \underline{0})$. Show that for $k \geq 128$ we have

$$\sup_{\theta \in \Theta_0} E_\theta \|\theta - \hat{\theta}\|^2 \geq (3/4)\sqrt{k} + 3, \quad \text{and}$$
$$\sup_{\theta \in \Theta_0} E_\theta \|\theta - \widetilde{\theta}\|^2 \leq 5.$$

Hint: A non-central χ_k^2 distribution is stochastically larger than a central χ_k^2 distribution; then use Lemma 1 of Laurent and Massart (2000).

6. **Optional bonus problem 2:** Problem 3.9, Lehmann and Casella, TPE, page 286.