

## Statistics 582, Problem Set 6

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**Due:** Wednesday, February 16, 2011.

**Reminder:** Midterm exam, Friday February 11.

**Reading:** Chapter 5, sections 7-8.

1. Consider Example 5.5.4 on pages 16 and 17 of the Chapter 5 notes.

(a) Show that the variance of  $\hat{\psi}$  is given by

$$\text{Var}(\hat{\psi}_n) = \frac{1}{n} \left\{ \frac{1}{B} \sum_{j=1}^B \frac{\theta_j}{\xi_j} - \psi(\theta)^2 \right\}.$$

[Hint: use the formula  $\text{Var}(Y) = E\text{Var}(Y|X) + \text{Var}[E(Y|X)]$  twice.]

(b) Use the result of (a) to show that

$$\text{Var}(\hat{\psi}_n) \leq \frac{1}{n\delta}$$

under the assumption that  $\xi_j \geq \delta > 0$  for all  $1 \leq j \leq B$ .

(c) What if the sampling probabilities  $\xi_i = 1$  for all  $i$ : do the conclusions of Example 4.5.4 and (a) and (b) above still hold?

2. A random variable  $X$  takes on values in the set  $\{1, 2, 3, 4\}$  with probability distributions  $p_0(x)$  or  $p_1(x)$  given in the following table.

x	1	3	3	4
$p_0(x)$	.1	.3	.4	.2
$p_1(x)$	.2	.2	.2	.4

(a) Find a most powerful test of size  $\alpha = .2$  for testing  $p_0$  versus  $p_1$  and determine its power.

(b) Find a test  $\phi$  which minimizes the sum of risks  $a + b$  where  $a \equiv E_0\phi$  and  $b = E_1(1 - \phi)$ .

(c) Compute  $d_{TV}(P_0, P_1)$ ,  $H(P_0, P_1)$ , and the affinity  $\rho(P_0, P_1)$ . For the product laws  $P_0^n$  and  $P_1^n$  (corresponding to observation of  $X_1, \dots, X_n$  i.i.d.  $P_0$  or  $P_1$  respectively), compute  $\rho(P_0^n, P_1^n)$  and  $H(P_0^n, P_1^n)$  for  $n = 10, 25, 100$ . What does this imply about the test  $\phi_n$  based on  $X_1, \dots, X_n$  which minimizes the sum of risks?

3. Suppose that  $X_1, \dots, X_n$  are i.i.d.  $N_k(\theta, \Sigma)$  with  $\Sigma$  known. Suppose that  $\theta \sim N_k(\mu, \tau^2 I)$ .
- (a) Find the Bayes estimator for estimating  $\theta$  with squared error loss  $L(\theta, a) = \|\theta - a\|^2 \equiv \sum_{j=1}^k (\theta_j - a_j)^2$ .
- (b) Use the result of (a) to show that  $\bar{X}_n$  is a minimax estimator of  $\theta$ .
4. **Optional bonus problem 1:** (a) Suppose that  $X \sim F$ , and let  $m = F^{-1}(1/2)$ ,  $\mu = E(X)$ ,  $\sigma^2 = Var(X)$ , and we assume that  $E(X^2) < \infty$ . Show that

$$|m - \mu| \leq \sqrt{2\sigma^2}.$$

Hint: use Chebychev's inequality.

- (b) Show that the inequality in (a) can be improved to  $|m - \mu| \leq \sigma$  by using convexity of the function  $\psi(x) \equiv |m - x|$  and concavity of  $\varphi(x) = \sqrt{x}$ .