

Statistics 582, Problem Set 4

Wellner; 1/26/2011

Reading: Chapter 5, sections 1-4.

Due: Wednesday, February 2, 2011.

- Let $\Theta = \{0, 1\} = \mathbf{A}$ where 0 = a patient has tuberculosis, 1 = a patient does not have tuberculosis. Let X be the number of positive reactions to two different tuberculosis tests, so that $\mathbf{X} = \{0, 1, 2\}$, and suppose that X has the following distributions

x	0	1	2
$p_0(x)$.02	.13	.85
$p_1(x)$.70	.27	.03

If the losses are given by $L(1, 1) = L(0, 0) = 0$, $L(0, 1) = 100$, $L(1, 0) = 10$, and the prior $\lambda = (\lambda_0, \lambda_1) = (.2, .8)$, find the Bayes rule d_B and the minimax rule d_M . Plot the risk set and label the non-randomized decision rules.

- Consider testing the simple hypothesis $H_0 : X \sim P_0$ versus the simple alternative $H_1 : X \sim P_1$. Let ϕ be a test of H_0 versus H_1 , and let $a \equiv E_1(1 - \phi)$, $b \equiv E_0\phi$.
 - Find a test ϕ which minimizes $a + Db$ where D is a fixed number.
 - When $D = 1$, relate the minimized total $a + b$ to the risk and to the total variation distance $d_{TV}(P_0, P_1)$ between P_0 and P_1 (or $\int p_0 \wedge p_1 d\mu$ for a dominating measure μ , e.g. $P_0 + P_1$).
 - Carry the computations of (b) through in the context of problem 1 when the losses are $L(0, 0) = L(1, 1) = 0$, $L(0, 1) = 10 = L(1, 0)$, and the prior distribution is $\lambda = (\lambda_0, \lambda_1) = (.5, .5)$.
- Let $\mathcal{X} = \{0, 1\}$, $\mathcal{A} = \Theta = \{1, 2\}$, and assume that the losses are given by $L(1, 1) = L(2, 2) = 0$, $L(1, 2) = a$, $L(2, 1) = b$. Suppose that the statistician can observe either X or Y where

$$\begin{aligned}
 p_1(1) &= P_1(X = 1) = 2/3, & p_2(1) &= P_2(X = 1) = 1/2, \\
 p_1^*(1) &= P_1(Y = 1) = 3/4, & p_2^*(1) &= P_2(Y = 1) = 1/2.
 \end{aligned}$$

Let $\underline{\lambda} = (\lambda, 1 - \lambda)$, $\lambda \in [0, 1]$ be the prior distribution over Θ .

- Find the Bayes risk when X is observed, and similarly for Y .
 - In the case $a = b$, $\lambda = 1/2$, would the statistician prefer to observe X or Y ?
 - For general $a \neq b$, $\lambda \in (0, 1)$ would the statistician prefer to observe X or Y ?
- Let $\Theta = \mathcal{A} = \{0, 1\}$, assume the (hypothesis testing) loss function $L(0, 0) = L(1, 1) = 0$, $L(0, 1) = L(1, 0) = 1$. Suppose that we observe the random variable X with the discrete distribution $P_\theta(X = x) = 2^{-(x+\theta)} 1_{\mathbb{Z}^+ \cap [1-\theta, \infty)}(x)$.
 - Describe the set of all non-randomized decision rules.
 - Plot the risk set \mathcal{R} in the plane. Which non-randomized rules are admissible? Why?

- (c) Can you find a non-randomized minimax rule?
 (d) What decision rules result from a Neyman-Pearson approach?
Hint: every number $z \in [0, 1]$ has a diadic representation of the form $\sum_{x=1}^{\infty} d(x)2^{-x}$ where $d(x) = 0$ or 1 .

5. Let X be a random variable with distribution function F having finite first moment: $E|X| < \infty$.

(a) Show that $f(b) \equiv E|X - b|$ is minimized by $b =$ any median of the distribution F of X . [A median m of F is any value satisfying $F(m) = P(X \leq m) \geq 1/2$ and $1 - F(m-) = P(X \geq m) \geq 1/2$; see Lehmann and Casella, TPE, page 62, problems 1.7 and 1.8.]

(b) For $0 < \tau < 1$, let $\rho_{\tau}(x) = x(\tau - 1_{(-\infty, 0)}(x))$. Consider minimizing

$$M_{\tau}(\theta) = E\rho_{\tau}(X - \theta)$$

with respect to θ . Show that the solution $\theta_0 = \theta_0(F)$ is given by the τ -th quantile of F : $\theta_0(F) = F^{-1}(\tau)$.

6. **Optional bonus problem:** Suppose that $\Theta = \{\theta_1, \theta_2\}$, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, and that the loss function $L(\theta, a)$ is given by the following table:

θ/a	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

Further suppose that $P_{\theta_j}(X = 0) = 1$ for $j = 1, 2$.

- (a) Find the decision risk set \mathcal{R} .
 (b) Find the decision rules that are Bayes with respect to the prior distribution $\lambda = (1, 0)$.
 (c) Show that the rule d_0 for which $R(\theta_1, d_0) = 1$ and $R(\theta_2, d_0) = 1$ is Bayes with respect to $\lambda = (1, 0)$ and also minimax, but that it is not admissible.
 (d) Relate this example to our theorem about admissibility of Bayes rules.