Statistics 582, Problem Set 9 Solutions

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1. Consider the Locally Most Powerful test ϕ for testing $H : \theta \le 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.
(a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.
(b) What is the behavior of each of these two approximations for large values of θ? Which of them shows that the power function decreases to 0 as $\theta \to \infty$? Why?
(c) Find a test ϕ of H versus K which has monotone increasing power function $\beta_\phi(\theta)$.

Solution: (a) The test is "reject H if $\sqrt{n}\,\overline{Y}_n > 2^{-1/2} z_\alpha$" where $Y_i \equiv 2X_i/(1+X_i^2)$ are i.i.d. and $X_i \sim$ Cauchy $(\theta, 1)$. Thus under P_θ, by using contour integration and Cauchy's formula, or by using Mathematica, Maple, or your favorite symbolic manipulation program,

$$\begin{aligned} m(\theta) \equiv E_\theta Y_i &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2} p_\theta(x)dx = \int_{-\infty}^{\infty} \frac{2x}{1+x^2}\frac{1}{\pi}\frac{1}{1+(x-\theta)^2}dx \\ &= \frac{2\theta}{4+\theta^2}, \end{aligned}$$

and

$$\begin{aligned} \sigma^2(\theta) &\equiv Var_\theta(Y_i) = E_\theta Y_i^2 - m^2(\theta) = \frac{2(4+3\theta^2)}{(4+\theta^2)^2} - \left(\frac{2\theta}{4+\theta^2}\right)^2 \\ &= \frac{2}{4+\theta^2}\,. \end{aligned}$$

For local alternatives $\theta = \theta_n = t/\sqrt{n}$, we have

$$\begin{aligned} \text{Power}(\theta_n) &= P_{\theta_n}(\sqrt{n}\,\overline{Y}_n > 2^{-1/2} z_\alpha) \\ &= P_{\theta_n}(\sqrt{n}(\overline{Y}_n - m(\theta_n)) \ge 2^{-1/2} z_\alpha - \sqrt{n}(m(\theta_n) - m(0))) \\ &\to P(2^{-1/2} Z \ge 2^{-1/2} z_\alpha - m'(0)t) \end{aligned}$$

where

$$\begin{aligned} m'(0) &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2}\frac{d}{d\theta} p_\theta(x)|_{\theta=0} dx \\ &= \int_{-\infty}^{\infty} \dot{l}_\theta(x;0)\dot{l}_\theta(x;0)p_\theta(x;0)dx = I(\theta) = 1/2\,. \end{aligned}$$

Hence we have

$$\text{Power}(\theta_n) \to P(Z > z_\alpha - 2^{-1/2}t) = 1 - \Phi(z_\alpha - 2^{-1/2}t)\,.$$

This approximation to the power function increases monotonically from α at $t = 0$ to 1 at $t = \infty$ (effectively when $t > 2^{1/2} \cdot 4$). Note that this result is very much in qualitative agreement with corollary 4.2.4 from Statistics 581.

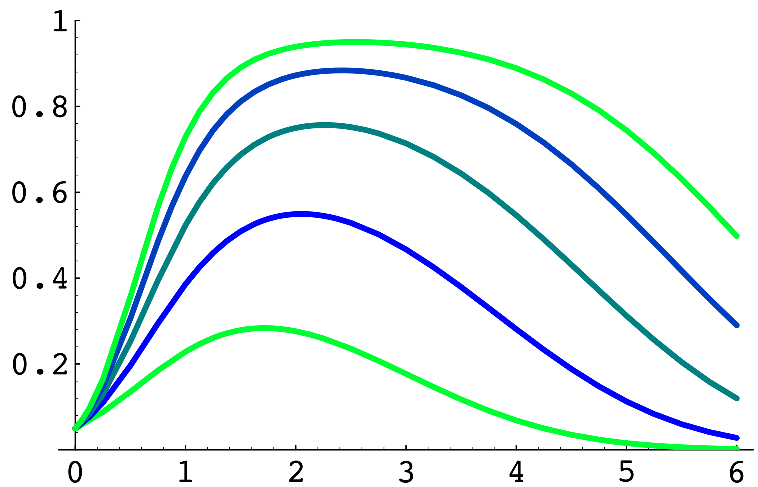

Figure 1: Plots of fixed θ power approximations for $n = 3, 6, 9, 12, 15$.

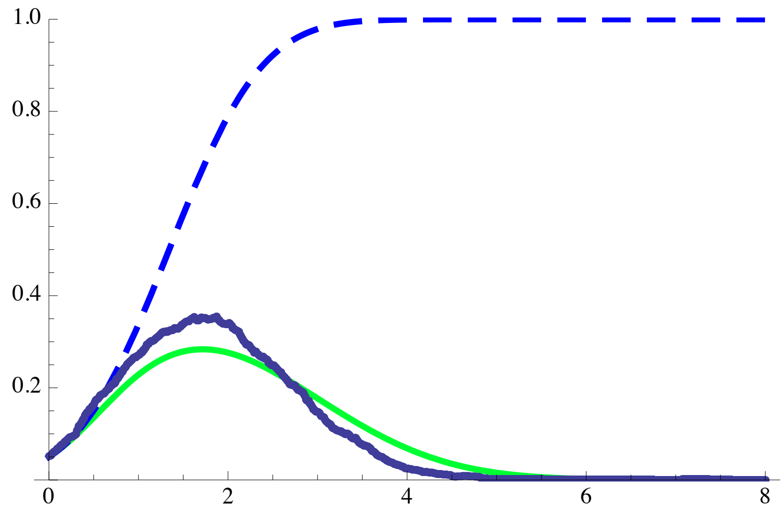

Figure 2: Local (dashed) and fixed (green) θ power approximations, $n = 3$; Monte Carlo estimate of true power (blue)

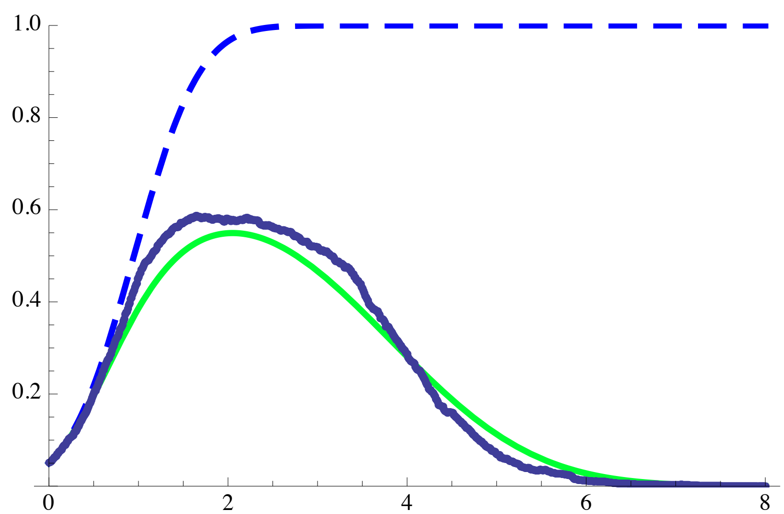

Figure 3: Local (dashed) and fixed (green) θ power approximations, $n = 6$; Monte Carlo estimate of true power (blue)

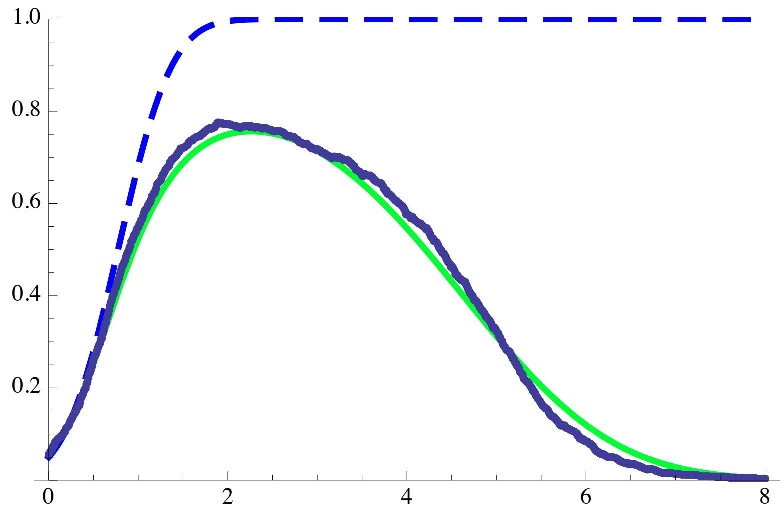

Figure 4: Local (dashed) and fixed (green) θ power approximations, $n = 9$; Monte Carlo estimate of true power (blue)

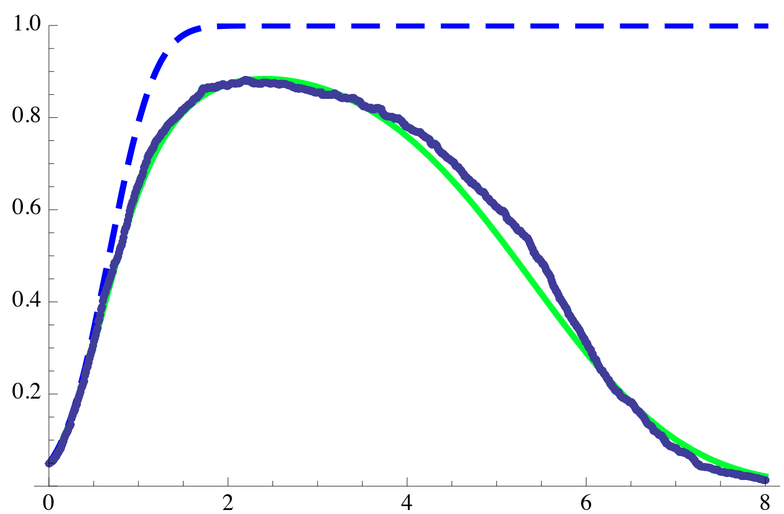

Figure 5: Local (dashed) and fixed (green) θ power approximations, $n = 12$; Monte Carlo estimate of true power (blue)

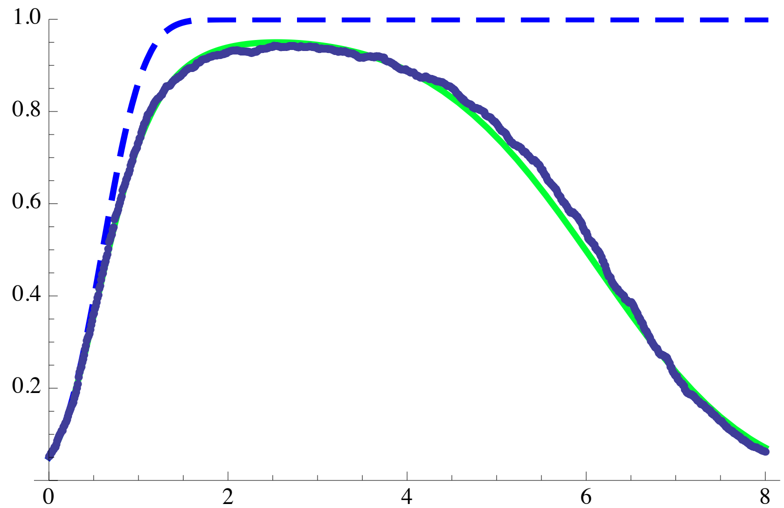

Figure 6: Local (dashed) and fixed (green) θ power approximations, $n = 15$; Monte Carlo estimate of true power (blue)

(b) For fixed alternatives $\theta > 0$ we have

$$\begin{aligned}
\text{Power}(\theta) &= P_\theta(\sqrt{n}\,\overline{Y} > 2^{-1/2} z_\alpha) \\
&= P_\theta(\sqrt{n}(\overline{Y}_n - m(\theta)) > 2^{-1/2} z_\alpha - \sqrt{n} m(\theta)) \\
&\doteq P(Z > (2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta)) \\
&= 1 - \Phi((2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta))\,.
\end{aligned}$$

where

$$\frac{m(\theta)}{\sigma(\theta)} = \frac{\sqrt{2\theta^2}}{\sqrt{4+\theta^2}}$$

which increases from 0 at $\theta = 0$ to a maximum of $1/\sqrt{2}$ at $\theta = 2$, and then tends to $\sqrt{2}$ as θ increases to ∞. Thus our fixed alternative approximation to the power function is completely determined by the function $(2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta)$, and this first decreases as θ increases (as it should if the power is to increase), but then it reaches a minimum and increases thereafter (approximately as $2^{-1} z_\alpha \theta - \sqrt{2n}$ as $\theta \to \infty$). Hence this approximation to the power decreases to zero just as we argued that it must in class. See Figures 1-6 which give plots of the two approximations for $\alpha = .05$ and $n = 3, 6, 9, 12, 15$, together with Monte-Carlo estimates of the true power function baed on $m = 1000$ Monte-Carlo replications at each sample size (dark blue). Note that the two approximations agree for θ's close to 0, but the local approximation is always monotone increasing, while the approximations with θ fixed show the approximate power decreasing to 0 as $\theta \to \infty$ as we know it must. Also note that fixed alternative approximation to the power function based on the CLT is reasonably accurate for $n \geq 9$.

(c) Consider $\overline{X}_n \equiv n^{-1}\sum_{i=1}^n X_i$. Then, since $X_i \stackrel{d}{=} V_i + \theta$ where V_i are i.i.d. standard Cauchy random variables with $Ee^{itV_1} = \exp(-|t|)$ (see e.g. Shorack, *Probability for Statisticians*, page 343), it follows that

$$\begin{aligned}
\phi_{\overline{X}_n}(t) &= E\exp(itn^{-1}\sum_1^n X_i) = \phi_{X_1}(t/n)^n = \big(E\exp(itn^{-1}X_1)\big)^n \\
&= \big(E\exp(itn^{-1}(V_1+\theta))\big)^n = \phi_{V_1}(t/n)^n \cdot e^{it\theta} = \\
&= \exp(-|t|)e^{it\theta} = \phi_{X_1}(t);
\end{aligned}$$

i.e. $\overline{X}_n \stackrel{d}{=} X_1$. Thus the test $\phi(\underline{X}) = 1\{\overline{X}_n > k\}$ has

$$\alpha = E_0\phi(\underline{X}) = P_0(\overline{X}_n > k = P_0(X_1 > k)$$

if $k = \tan((1/2-\alpha)\pi) \equiv k_\alpha$, and then

$$\begin{aligned}
\beta_\phi(\theta) &= P_\theta(\overline{X}_n > k_\alpha) = P_\theta(X_1 - \theta > k_\alpha - \theta) \\
&= P_0(V_1 > k_\alpha - \theta) = \int_{k_\alpha-\theta}^\infty \frac{1}{\pi}\frac{1}{1+y^2}dy
\end{aligned}$$

which is monotone increasing as a function of θ; see Figure 7. On the other hand, this test does not yield increasing power as the sample size n increases: it has the same power function for all n!

Here is another possibility: Let $Z_i \equiv 1\{X_i > 0\}$ for $i = 1, \ldots, n$. Thus the Z_i's are independent Bernoulli $p = p_\theta$ random variables with

$$\begin{aligned}
p_\theta &= P_\theta(X_i > 0) = P_\theta(X_i - \theta > -\theta) = P_0(Y_i > -\theta) = \int_{-\theta}^\infty \frac{1}{\pi}\frac{1}{1+y^2}dy \\
&\nearrow 1 \ \text{ as } \theta \to \infty.
\end{aligned}$$

Note that testing $H : \theta \leq \theta_0 = 0$ versus $K : \theta > 0$ is equivalent to testing $H' : p \leq p_0 \equiv 1/2$ versus $K' : p > p_0 = 1/2$. Now

$$T_n \equiv \sum_1^n Z_i = n(1 - \mathbb{F}_n(0)) \sim \text{Binomial}(n, p)$$

is a one-parameter exponential family with density function $p(\cdot;p)$ with respect to counting measure on $\mathbb{N}$ given by

$$\begin{aligned} p(y;p) &= P_p(T_n = y) = \binom{n}{y} p^y (1-p)^{n-y} = \binom{n}{y} \left(\frac{p}{1-p}\right)^y (1-p)^n \\ &= c(p)\exp(Q(p)y)h(y) \end{aligned}$$

with $Q(p) \equiv \log(p/(1-p))$, $c(p) \equiv (1-p)^n$, and $h(y) = \binom{n}{y}$, so it has monotone likelihood ratio in y. Thus by the Karlin-Rubin theorem, the test

$$\varphi(\underline{X}) = \phi(\underline{Z}) = \begin{cases} 1 & \text{if } T_n = \sum_1^n Z_i > k \\ \gamma & \text{if } T_n = k \\ 0 & \text{if } T_n < k, \end{cases}$$

where k and γ are chosen so that $E_{p_0=1/2}\phi(\underline{Z}) = \alpha$, has power function

$$\beta_\phi(p_\theta) \equiv \tilde{\beta}_\varphi(\theta)$$

which is monotone increasing in p_θ and θ. This test improves with increasing sample size n; see Figure 7

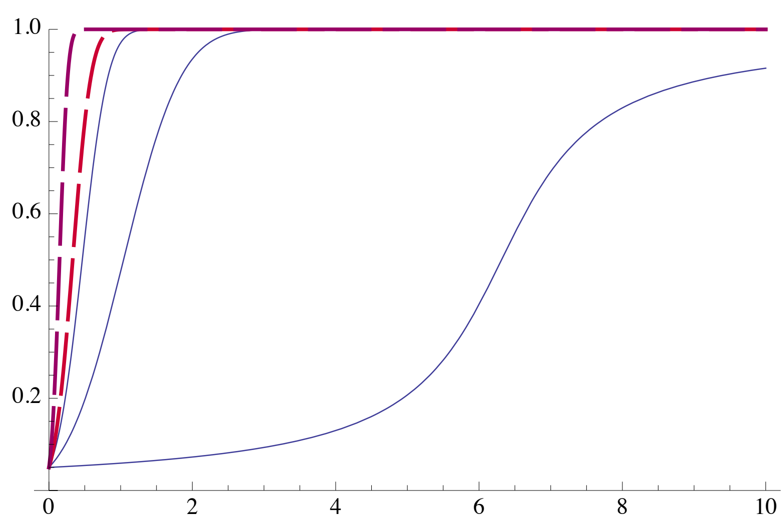

Figure 7: Plot comparing local power approximations for the Binomial test (solid), locally most powerful test (dashed), and the mean test, for $n = 10,\ 50$.

2. Let $X_1, \ldots, X_n$ be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
(a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.
(b) Find the subclass of the tests that are independent of θ_1. These are UMP tests of H_0 versus $H_1' : \theta > \theta_0$.
(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is UMP of size α for testing $H_0' : \theta \le \theta_0$ versus $H_1' : \theta > \theta_0$ but that ϕ is not admissible.
(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \le b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \ne \theta_0$.

Solution: (a) For testing $\theta = \theta_0$ versus $\theta = \theta_1 > \theta_0$, the class of all NP tests is given by tests of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) > k \theta_0^{-n} 1_{[0,\theta_0]}(t) \\ \gamma(t), & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) = k \theta_0^{-n} 1_{[0,\theta_0]}(t) \\ 0, & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) < k \theta_0^{-n} 1_{[0,\theta_0]}(t) \end{cases} .$$

Thus for $k = (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t), & \text{if else ;} \end{cases}$$

for $k > (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t), & \text{if else} \\ 0, & \text{if } 0 \le t \le \theta_0; \end{cases}$$

and for $k < (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } 0 \le t \le \theta_1 \\ \gamma(t), & \text{if else.} \end{cases}$$

(b) The subclass of these tests that do not depend on θ_1 is the class of tests ϕ with

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t < \infty \\ \gamma(t), & \text{if else} \end{cases}$$

where $E_{\theta_0} \gamma(T) = \alpha$.
(c) The test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is of the form of the tests in (b) with $\gamma(t) = \alpha$ for $0 \le t \le \theta_0$, and hence is UMP of size α for testing $\theta = \theta_0$ versus $\theta > \theta_0$. To see that it is UMP for testing $\theta \le \theta_0$ versus $\theta > \theta_0$, we first compute

its power function to confirm that it is of size α for the composite null hypothesis $\theta \le \theta_0$. The power is

$$\begin{aligned}
\beta_\phi(\theta) &= E_\theta \phi(T) = P_\theta(T > \theta_0) + \alpha P_\theta(T \le \theta_0) \\
&= \{1 - (\frac{\theta_0}{\theta})^n\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta) + \alpha \left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0,\infty)}(\theta) \\
&= \{1 - (\frac{\theta_0}{\theta})^n (1-\alpha)\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta).
\end{aligned}$$

Thus we see that $\beta_\phi(\theta) = \alpha$ for $\theta \le \theta_0$, and ϕ is of size α for $\theta \le \theta_0$. Since the class of size α tests for testing $\theta = \theta_0$ is a larger class than the class of size α tests for testing $\theta \le \theta_0$, and since ϕ is UMP in the larger class, it is also UMP in the smaller class. Hence ϕ is UMP for testing $\theta \le \theta_0$ versus $\theta > \theta_0$. But the competing test $\phi_0(t) = 1\{t > (1-\alpha)^{1/n}\theta_0\}$ has power function

$$\begin{aligned}
\beta_{\phi_0}(\theta) &= E_\theta \phi_0(T) = P_\theta(T > \theta_0(1-\alpha)^{1/n}) \\
&= 1 - \{(1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0(1-\alpha)^{1/n},\infty)}(\theta) + 1_{[0,\theta_0(1-\alpha)^{1/n}]}(\theta)\},
\end{aligned}$$

so the power function of the test ϕ_0 is strictly below that of the test ϕ on the set $[0, \theta_0)$. Hence ϕ is inadmissible and the test ϕ_0 is also UMP.
(d) The test $\phi(t) = 1 - 1_{(\theta_0\alpha^{1/n},\theta_0]}(t)$ is of size α for testing $\theta = \theta_0$ versus $\theta \ne \theta_0$ since

$$E_{\theta_0}\phi(T) = P_{\theta_0}(T \le \theta_0 \alpha^{1/n}) = \left(\frac{\theta_0\alpha^{1/n}}{\theta_0}\right)^n = \alpha.$$

Furthermore it is of the form of the class of all UMP tests for testing $\theta = \theta_0$ versus $\theta > \theta_0$, hence it UMP among the size α tests for $\theta > \theta_0$. For testing $\theta = \theta_0$ versus $\theta = \theta_1 < \theta_0$, the NP Pearson tests of the form $\phi_1(t) = \gamma(t)1_{[0,\theta_0]}(t)$ are most powerful of their size. But the test ϕ is of this form (with $\gamma(t) = 1_{[0,\theta_0\alpha^{1/n}]}(t)$), is in this class and does not depend on $\theta_1 < \theta_0$. Hence ϕ is UMP for testing $\theta = \theta_0$ versus $\theta \ne \theta_0$. [This is an unusual situation in which we get "something for free" from the structure of the uniform distributions. Usually two-sided tests are *not* UMP!]

Determine whether there exists a level-α test of $H: \ P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

3. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 are the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \ P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

Solution: Here the table of likelihood ratios is as follows:

x	1	2	3	4	5	6
$p_1(x)/p_0(x)$	2	5/2	4	2	∞	78/98
$p_2(x)/p_0(x)$	3	5/2	6	0	∞	72/98

(i) For $\alpha = .01$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1\{x = 5\} + (1/2)1\{x = 3\},$$
$$\phi_2(x) = 1\{x = 5\} + 1/2)1\{x = 3\},$$

so $\phi_1 = \phi_2$ is Uniformly most powerful.
(ii) For $\alpha = .05$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,3,5\}},$$

so there is no UMP test of P_0 versus P_1 and P_2 at this level.
(iii) For $\alpha = .07$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,2,3,5\}},$$

so by taking $\gamma(x) = 1\{x = 1\}$, $\phi_1(x) = \phi_2(x)$, and this test is Uniformly Most Powerful for testing P_0 versus P_1 and P_2.

4. For observations $\underline{X} = (X_1, \ldots, X_n)$, let $X_{(1)} \le \ldots \le X_{(n)}$ denote the *order statistics* of the X_i's ($X_{(i)} \equiv \mathbb{F}_n^{-1}(i/n)$, $i = 1, \ldots, n$) and let $\underline{R} = (R_1, \ldots, R_n)$ denote the *ranks*; defined by $X_i = X_{(R_i)}$, $i = 1, \ldots, n$ (if $X_i = X_j$ for some $i < j$, define the ranks by $R_i < R_j$ and $X_i = X_{(R_i)}$).
(a) Suppose that $X_1, \ldots, X_n$ are i.i.d. $F \in \mathcal{F}_{ac}$ (the absolutely continuous df's F on R) with density f. Show that the order statistics $\underline{X}_{(\cdot)} \equiv (X_{(1)}, \ldots, X_{(n)})$

are independent of the ranks $\underline{R}$ and that the order statistics have joint density $\bar{p}$ given by

$$\bar{p}(\underline{x}_{(\cdot)}) = n! \prod_{i=1}^{n} f(x_{(i)}) \, , \qquad -\infty < x_{(1)} < \ldots < x_{(n)} < \infty$$

while

$$P(\underline{R} = \underline{r}) = \frac{1}{n!} \, , \qquad \underline{r} \in \Pi \equiv \{ \text{ all permutations of } \ \{1, \ldots, n\}\} \ .$$

(b) Show that (a) continues to hold for any joint distribution p of the $\underline{X}$ which is symmetric with respect to permutation of its coordinates: $p(\pi \underline{x}) = p(\underline{x})$ for all $\underline{x}$ and $\pi \in \Pi$ where $\pi x \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)})$.
(c) If the joint distribution p of $\underline{X}$ is general (not permutation symmetric), show that the joint density $\bar{p}$ of the order statistics is given by

$$\bar{p}(\underline{x}_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}) \ ,$$

and

$$P(\underline{R} = \underline{r} | \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r \underline{x}_{(\cdot)})}{\bar{p}(\underline{x}_{(\cdot)})} \ .$$

Solution: I will prove (c) first; then (a) and (b) follow as corollaries:
(c) Suppose that $\underline{X}$ has joint density p. Then for any set Borel set $A \subset \{\underline{x} \in \mathbb{R}^n : \ x_1 < x_2 < \cdots < x_n\}$

$$\begin{aligned} P(\underline{X}_{(\cdot)} \in A) &= \int_{[\underline{x}_{(\cdot)} \in A]} p(x_1, \ldots, x_n) \ dx_1 \ldots dx_n \\ &= \sum_{r \in \Pi} \int_{[R(\underline{x}) = r, \ \underline{x}_{(\cdot)} \in A]} p(x_1, \ldots, x_n) \ dx_1 \ldots dx_n \\ &= \sum_{r \in \Pi} \int_A p(x_{(r_1)}, \ldots, x_{(r_n)}) \ dx_{(1)} \ldots dx_{(n)} \\ &= \int_A \overline{p}(x_{(1)}, \ldots, x_{(n)}) \ dx_{(1)} \ldots dx_{(n)} \end{aligned}$$

where we have used the fact that the correspondence between $(x_1, ..., x_n)$ and $(x_{(1)}, \ldots, x_{(n)})$ is one-to-one and linear with Jacobian $= 1$ on each subset $[R = r]$, $r \in \Pi$. This proves that

$$\overline{p}(\underline{x}_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}) \ .$$

Similarly,

$$\begin{aligned} P(R = r,\ \underline{X}_{(\cdot)} \in A) &= \int_{[R=r,\ \underline{x}_{(\cdot)} \in A]} p(x_1, \ldots, x_n)\ dx_1 \cdots dx_n \\ &= \int_A p(x_{(r_1)}, \ldots, x_{(r_n)})\ dx_{(1)} \ldots dx_{(n)} \\ &= \int_A \frac{p(x_{(r_1)}, \ldots, x_{(r_n)})}{\overline{p}(x_{(1)}, \ldots, x_{(n)})} \overline{p}(x_{(1)}, \ldots, x_{(n)})\ dx_{(1)} \ldots dx_{(n)} \end{aligned}$$

since $\overline{p}(x_{(1)}, \ldots, x_{(n)}) = 0$ implies $p(x_{r(1)}, \ldots, x_{r(n)}) = 0$ for each $r \in \Pi$. This implies that

$$P(\underline{R} = \underline{r} | \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r\underline{x}_{(\cdot)})}{\overline{p}(\underline{x}_{(\cdot)})}\ .$$

(b) When $p(\underline{x}) = p(\pi \underline{x})$ for all $\pi \in \Pi$, then

$$\overline{p}(\underline{x}_{(\cdot)}) = n! p(\underline{x}_{(\cdot)}),$$

and

$$P(\underline{R} = \underline{r} | \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r\underline{x}_{(\cdot)})}{\overline{p}(\underline{x}_{(\cdot)})} = \frac{p(r\underline{x}_{(\cdot)})}{n! p(\underline{x}_{(\cdot)})} = \frac{1}{n!}.$$

Hence R is independent of $\underline{X}_{(\cdot)}$, and $P(R = r) = 1/n!$ for each $r \in \Pi$.
(a) This follows easily from (b) since, in this case, for any permutation π

$$p(\pi \underline{x}) = \prod_{i=1}^{n} f(x_{\pi(i)}) = \prod_{i=1}^{n} f(x_i) = p(\underline{x}).$$