Statistics 582, Problem Set 9

Wellner; 3/3/2010

Reading: Chapter 6, sections 6.1 and 6.2 (through page 19).
Due: Wednesday, March 10, 2010.
Reminder: Final Exam, Monday, March 15, 8:30 - 10:30 (Balmer 411)

1. Consider the Locally Most Powerful test ϕ for testing $H : \theta \le 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.
(a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.
(b) What is the behavior of each of these two approximations for large values of θ? Which of them shows that the power function decreases to 0 as $\theta \to \infty$? Why?
(c) Find a test ϕ of H versus K which has monotone increasing power function $\beta_\phi(\theta)$.

2. Let $X_1, \ldots, X_n$ be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
(a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.
(b) Find the subclass of the tests that are independent of θ_1. These are UMP tests of H_0 versus $H_1' : \theta > \theta_0$.
(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is UMP of size α for testing $H_0' : \theta \le \theta_0$ versus $H_1' : \theta > \theta_0$ but that ϕ is not admissible.
(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \le b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \ne \theta_0$.

3. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \; P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

4. For observations $\underline{X} = (X_1, \ldots, X_n)$, let $X_{(1)} \le \ldots \le X_{(n)}$ denote the *order statistics* of the X_i's ($X_{(i)} \equiv \mathbb{F}_n^{-1}(i/n)$, $i = 1, \ldots, n$) and let $\underline{R} = (R_1, \ldots, R_n)$ denote the *ranks*; defined by $X_i = X_{(R_i)}$, $i = 1, \ldots, n$ (if $X_i = X_j$ for some $i < j$, define the ranks by $R_i < R_j$ and $X_i = X_{(R_i)}$).
(a) Suppose that $X_1, \ldots, X_n$ are i.i.d. $F \in \mathcal{F}_{ac}$ (the absolutely continuous df's F on R) with density f. Show that the order statistics $\underline{X}_{(\cdot)} \equiv (X_{(1)}, \ldots, X_{(n)})$ are independent of the ranks $\underline{R}$ and that the order statistics have joint density $\bar{p}$ given by

$$\bar{p}(\underline{x}_{(\cdot)}) = n! \prod_{i=1}^{n} f(x_{(i)}) , \qquad -\infty < x_{(1)} < \ldots < x_{(n)} < \infty$$

while

$$P(\underline{R} = \underline{r}) = \frac{1}{n!} , \qquad \underline{r} \in \Pi \equiv \{ \text{ all permutations of } \{1, \ldots, n\}\} .$$

(b) Show that (a) continues to hold for any joint distribution p of the $\underline{X}$ which is symmetric with respect to permutation of its coordinates: $p(\pi \underline{x}) = p(\underline{x})$ for all $\underline{x}$ and $\pi \in \Pi$ where $\pi x \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)})$.
(c) If the joint distribution p of $\underline{X}$ is general (not permutation symmetric), show that the joint density $\bar{p}$ of the order statistics is given by

$$\bar{p}(\underline{x}_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}) ,$$

and

$$P(\underline{R} = \underline{r} | \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r\underline{x}_{(\cdot)})}{\bar{p}(\underline{x}_{(\cdot)})} .$$

5. **Optional bonus problem 1:** (Problem 3.7, Lehmann and Romano, TSH, page 94.) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	θ	2θ	$.9 - 2\theta$	$.1 - \theta$

where $0 < \theta < .1$. For testing $H: \theta = .05$ against $\theta > .05$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = .5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

6. **Optional bonus problems 2:** (a) Show that the logistic distribution with location parameter θ having density

$$p_\theta(x) = \frac{\exp(x-\theta)}{(1+\exp(x-\theta))^2} = \frac{1}{2(1+\cosh(x-\theta))}$$

has monotone likelihood ratio.
(b) Unfortunately the result of (a) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ as in (a), then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

7. **Optional bonus problem 3:** (From the material on consistency of Neyman Pearson tests, section 6.1; and Donoho and Jin, *Ann. Statist.* **32** (2004), 962-994)
Let $p_\mu(x) \equiv \phi(x-\mu)$ denote the density of $X \sim N(\mu, 1)$, and let P_μ denote the corresponding measure on $\mathbb{R}$.
(a) Consider testing $H: \ P = P_0$ versus $K: \ Q = P_\mu$ with $\mu > 0$ fixed. Compute $\rho(P,Q) = \rho(P_0, P_\mu) = \int \sqrt{p_0(x)p_\mu(x)}dx$ explicitly as a function of μ.
(b) Compare the power functions of the following two tests of H versus K when $\mu = 1$:
(i) The Neyman - Pearson test with $\alpha = .05$; (ii) The Neyman - Pearson type test with $k = k_n = 1$ (in the notation of Theorem 6.1.4, page 8, Chapter 6).
(c) Now suppose $\mu = \mu_n \equiv t/\sqrt{n}$ with $t > 0$ and consider testing $H : P = P_0$ versus $K_n : \ P = P_{\mu_n}$.
(i) Find the limit of $\rho(P_0, P_{\mu_n})^n$ as a function of t.
(ii) Compare the limiting power function of the two tests in (b).
(d) Now suppose that $Q = Q_n$ is given by the mixture

$$q(x) = q(x; \mu, \epsilon) = (1-\epsilon)p_0(x) + \epsilon p_\mu(x)$$

where $\epsilon = \epsilon_n = n^{-\beta}$ with $1/2 < \beta < 1$ and $\mu = \mu_n = \sqrt{2r \log n}$ with $r > \rho^*(\beta)$ and where the function

$$\rho^*(\beta) = \begin{cases} \beta - 1/2, & 1/2 \le \beta < 3/4, \\ (1-\sqrt{1-\beta})^2, & 3/4 \le \beta < 1. \end{cases}$$

Show that k_n can be chosen so that $E_{P_0^n}\phi_n(\underline{X}) \to 0$ and $E_{Q_n}^n(1-\phi_n(\underline{X}_n)) \to 0$.