

Statistics 582, Problem Set 8

Wellner; 2/24/2010

Reading: Chapter 6, sections 6.1 and 6.2 (through page 19);

Lehmann and Romano, Chapter 3.

Due: Wednesday, March 3, 2010.

1. Lehmann and Casella, problem 5.4, page 289 (with a modification to the posterior variance formula):
 Albert and Gupta (1985) investigate theory and applications of the hierarchical model

$$\begin{aligned} X_i | \boldsymbol{\theta}_i &\sim \text{Binomial}(n, \boldsymbol{\theta}_i), \quad i = 1, \dots, p, \quad \text{independent} \\ \boldsymbol{\theta}_i | \eta &\sim \text{Beta}(k\eta, k(1 - \eta)), \quad k \text{ known,} \\ \eta &\sim \text{Uniform}(0, 1). \end{aligned}$$

- (a) Show that

$$\begin{aligned} E(\boldsymbol{\theta}_i | \underline{X}) &= \left(\frac{n}{n+k} \right) \frac{X_i}{n} + \frac{k}{n+k} E(\eta | \underline{X}), \\ \text{Var}(\boldsymbol{\theta}_i | \underline{X}) &= \frac{k^2}{(n+k)(n+k+1)} \text{Var}(\eta | \underline{X}) + \frac{E(\boldsymbol{\theta}_i | \underline{X})(1 - E(\boldsymbol{\theta}_i | \underline{X}))}{n+k+1}. \end{aligned}$$

- (b) Show that the $\boldsymbol{\theta}_i$'s have posterior covariance

$$\text{Cov}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j | \underline{X}) = \left(\frac{k}{n+k} \right)^2 \text{Var}(\eta | \underline{X}), \quad i \neq j.$$

- (c) Ignoring the prior distribution of η , show how to construct an empirical Bayes estimator of $\boldsymbol{\theta}_i$. (This might not be expressible in simple closed form.)

2. Suppose that $X_i \sim \text{Binomial}(n, \theta_i)$ are independent for $i = 1, \dots, k$ conditionally on $\boldsymbol{\theta}_i \sim \text{Beta}(a, b)$. Let $L(\underline{\boldsymbol{\theta}}, \underline{\boldsymbol{a}}) = \sum_{i=1}^k (\theta_i - a_i)^2$.

(a) Find the Bayes rules $d(\underline{X}) \equiv (d_i(X_i) = d_i(X_i; a, b), i = 1, \dots, k)$ for estimation of $\boldsymbol{\theta}_i$ for $i = 1, \dots, k$, assuming that $a, b > 0$ are known.

(b) Find the marginal distribution (or mass function) of $\underline{X} = (X_1, \dots, X_k)$.

(c) Consider estimation of the parameters a, b by maximizing the marginal likelihood resulting from part (b). Denote the resulting estimators of a, b by \hat{a}, \hat{b} . (These will not be expressible in closed form in general.)

(d) Now consider the *empirical Bayes* rules

$$\hat{d}(\underline{X}, \hat{a}, \hat{b}) \equiv (d_i(X_i, \hat{a}, \hat{b}), i = 1, \dots, k).$$

(e) Compare the Bayes risks of $d(\underline{X})$, $\hat{d}(\underline{X}, \hat{a}, \hat{b})$, and \underline{X}/n when $a = b = 2$, $n = 20$, and $k = 10$. [Hint: see Lehmann and Casella, pages 263-266, including Table 6.1 and Theorem 6.3 on page 265, and (6.16) on page 266.]

3. Suppose that X_1, \dots, X_n are i.i.d. $N(\theta, \sigma^2)$.
- (a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ . Find the entire risk body and plot it.
 - (b) What happens to the risk body as n grows or as $\sigma_0 \rightarrow 0$?
 - (c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
 - (d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : n \geq 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?
4. A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$.18	.06	.36	.40
$p_1(x)$.36	.18	.24	.22

- (a) Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
 - (b) Find a Bayes rule ϕ for testing $H_0 : p = p_0$ versus $H_1 : p = p_1$ when the prior distribution is given by $(\lambda, 1 - \lambda) = (1/2, 1/2)$. What is the relationship between the Bayes rule and the rule which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$?
5. **Bonus problem 1:** Show that the logistic distribution with location parameter θ having density

$$p_\theta(x) = \frac{\exp(x - \theta)}{(1 + \exp(x - \theta))^2} = \frac{1}{2(1 + \cosh(x - \theta))}$$

has monotone likelihood ratio.