Statistics 582, Problem Set 7, corrected

Wellner; 2/17/2010

Reading: Chapter 5, sections 5.7-5.8;
start reading Chapter 6 (to be handed out Monday 2/22)

Due: Wednesday, February 24, 2010.

1. Continuation of problem 4, problem set 5: Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have distribution P_θ and density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on $\mathbb{R}$, and that $\theta \sim \Gamma(\alpha, \beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

In problem set 5 we found the Bayes rules with respect to squared error loss $L(\theta, a) = (\theta - a)^2$ and weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$.

(a) Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
(b) What does theorem 5.8.2 say about the limiting distribution of the Bayes rule for squared error loss (assuming that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \equiv P$ with $\theta_0 \in (0, \infty)$)?

2. Let $\Theta = (0, \infty)$, $\mathbf{A} = [0, \infty)$, let X have the discrete distribution

$$p(x, \theta) = \binom{r + x - 1}{x} \theta^x (\theta + 1)^{-(r+x)}, \qquad x = 0, 1, 2, \ldots$$

where r is some known positive integer; this is the negative binomial distribution reparametrized so that $E_\theta X = r\theta$. Suppose that

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta(\theta + 1)}.$$

(a) Show that the usual estimator, $d_0(X) = X/r$ is an equalizer rule (i.e. a decision rule such that $R(\theta, d)$ is constant in θ).
(b) Show that the usual estimator d_0 is generalized Bayes with respect to Lebesgue measure on $(0, \infty)$ provided $r > 1$. (What happens if $r = 1$?) (An estimator is called a "generalized Bayes rule" if the posterior distribution (and any conditional expectations needed to define the Bayes rule) is (are) well defined, even though the prior distribution is not a "proper" prior with total mass 1.)

(c) Find Bayes decision rules with respect to the prior distributions $\Lambda_{\alpha,\beta}$ with densities

$$\lambda_{\alpha,\beta}(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(\theta+1)^{-(\alpha+\beta)}1_{(0,\infty)}(\theta),$$

the distribution of $\theta = Z/(1-Z)$ where $Z \sim$ Beta(α,β).
(d) Show that $d(X) = X/(r+1)$ is minimax. [Note that d_0 is not minimax, hence not admissible.]

3. Specialize the decision rule in Theorem 5.2 of the course notes to the case when P_i is the normal distribution $N_d(\mu_i, I)$, $i = 1, \ldots, k$ where $\mu_1, \ldots, \mu_k$ are distinct vectors in $\mathbb{R}^d$, $\mu_i \neq \mu_j$ for $i \neq j$. What happens if we replace I by Σ?

4. Suppose that $X_n \equiv X \sim$ Multinomial$_k(n, \underline{\theta})$.
(a) Suppose that the prior distribution on θ is given by a Dirichlet distribution, Dirichlet$(\underline{\alpha})$:

$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)}\theta_1^{\alpha_1-1}\cdots\theta_k^{\alpha_k-1}1_{[\underline{\theta}:\ \sum\theta_i=1]}\,.$$

Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4
(b) What is the posterior distribution for θ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
(c) Find a minimax estimator d_M of $\underline{\theta}$.

5. Find the limit distribution of the minimax estimator d_M in problem 4 (i.e. $\sqrt{n}(d_M(X_n) - \theta) \to_d$ "something" and find "something"). Is d_M a regular estimator of θ?

6. **Optional bonus problem 1:** Problem 3.9, Lehmann and Casella, TPE, page 286.

7. **Optional bonus problem 2:** Problem 5.17, Lehmann and Casella, TPE, page 293.