Statistics 582, Problem Set 2

Wellner; 1/11/2010

Reading: Lehmann and Casella, TPE, Chapter 6, section 6.4,
especially pages 455 - 461 and 504-508.
Chapter 4, sections 5 - 6.

Due: Wednesday, January 20, 2010.

1. Problem 1, page 117, Ferguson, ACILST. What happens if $\Theta = [1, \infty)$ or $(0, \infty)$?

2. (a) Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with distribution P on $\mathbb{R}$ satisfying $E|X| < \infty$. Consider the example discussed in class on 1/8/2010: if V_n is defined by
$$V_n \equiv \frac{1}{n}\sum_{i=1}^n |X_i - \overline{X}_n| = \mathbb{P}_n|X - \overline{X}_n|,$$
show that
$$V_n \rightarrow_{a.s.} v \equiv E|X - \mu|$$
where $\mu = E(X)$.
(b) Now suppose we generalize the problem considered in (a) by considering $X_1, \ldots, X_n$ i.i.d. P on $\mathbb{R}^d$ and powers other than 1: let $\|\cdot\|$ be the usual Euclidean metric in $\mathbb{R}^d$, and consider
$$V_n(r) \equiv \frac{1}{n}\sum_{i=1}^n \|X_i - \overline{X}_n\|^r$$
for $1 \le r \le 2$ where $\overline{X}_n$ is the (multivariate) sample mean of the X_i's. Show that if $E|X|^r < \infty$, then $V_n(r) \rightarrow_{a.s.} v(r)$ where $v(r) \equiv E\|X_1 - \mu\|^r$ where $\mu = E(X) \in \mathbb{R}^d$?

3. (a) In connection with Problem 1.3 (Ferguson, ACILST, page 117, problem 2, with parameter space $\Theta = [0, 1]$), does Wald's Theorem 4.3, page 28, Chapter 4 notes apply?
(b) What happens if $\Theta = [0, \infty)$?
(c) Do our hypotheses A0-A2 (of Section 4.1) hold in this example?
(d) Do our hypotheses A3 and A4 (of Section 4.1) hold in this example?
(e) Does there exist an estimator $\bar{\theta}_n$ of θ which is $n^{1/2}-$consistent?

4. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. Weibull(α_0, β_0) (if X has the Weibull(θ) distribution where $\theta = (\alpha, \beta)$, then $1 - F_\theta(x) = P_\theta(X > x) = \exp(-(x/\alpha)^\beta)$ for $x \ge 0$). Recall that the MLE $\widehat{\alpha}$ of α is given by
$$\widehat{\alpha} = \left\{ \frac{1}{n}\sum_{i=1}^n X_i^{\widehat{\beta}} \right\}^{1/\widehat{\beta}}$$

where $\widehat{\beta}$ is the MLE of β. As a simpler alternative to maximum likelihood, I propose to use the alternative estimator $\overline{\beta}_n$ of β obtained from the slope of an ordinary least squares fit of a Weibull Q-Q plot, and then estimate α by

$$\overline{\alpha}_n = \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^{\overline{\beta}_n} \right\}^{1/\overline{\beta}_n} .$$

(a) Suppose that $\overline{\beta}_n \to_p \beta_0$ is known. Show that $\overline{\alpha}_n \to_p \alpha_0$. [Hint: use a uniform strong law of large numbers.]
(b) Show that $\overline{\alpha}_n$ is a "pseudo-MLE" in the sense that $\overline{\alpha}_n$ maximizes $l_n(\alpha, \overline{\beta}_n)$.
(c) What hypotheses are needed if we assume $X_1, \ldots, X_n$ are i.i.d. P_0 with P_0 not necessarily Weibull? You may continue to assume that $\overline{\beta}_n \to_p \beta \equiv \beta(P_0)$.

5. On pages 116-117 of ACILST, Ferguson (see also Ferguson, T. S. (1982). An inconsistent maximum likelihood estimate. *J. Amer. Statist. Assoc.* **77**, 831–834) shows that $\hat{\theta}_n \to_{a.s.} 1$ no matter what θ_0 is true if $\delta(\theta) \to 0$ "fast enough".
(a) Show that $\hat{\theta}_n \to_{a.s.} 1$ continues to hold if

$$\delta(\theta) = (1-\theta)\exp(-(1-\theta)^{-c} + 1)$$

with $c > 2$. (Ferguson shows that $c = 4$ works.)
(b) Show that when $c = 2$, Ferguson's argument yields

$$\sup_{0 \le \theta \le 1} n^{-1} \log L_n(\theta) \ge \frac{n-1}{n} \log(M_n/2) + \frac{1}{n} \log \frac{1-M_n}{\delta(M_n)} \to_d D$$

where

$$P(D \le y) = \exp\left(-\frac{1}{2(y - \log 2)} \right), \quad y \ge \log(2) .$$

That is, $D \stackrel{d}{=} \log 2 + 1/(2E)$ where E is an Exponential(1) random variable.

6. **Optional bonus problem.** This is a continuation of problem 5 above.
(c) Show that the hypothesis $EF(X) < \infty$ fails in this example with $c = 4$; does it also fail with $c = 2$?

7. **Optional bonus problem:** In the context of problem 1 above, and with the added assumption that $E|X|^2 < \infty$, show that $\sqrt{n}(V_n - v) \to N(0, C^2)$ under some modest further assumption and find C^2. (What tools do you need now?)

8. **Optional bonus problem:** Consider Problem 3 above with $\Theta = [0, \infty)$, and suppose that $\theta_0 \in (0, \infty)$. Find the limiting distribution of the MLE $\widehat{\theta}_n$.