Statistics 582, Problem Set 1

Wellner; 1/4/2010

Reading: Chapter 4, Sections 4-6; Lehmann and Casella, TPE, section 6.6, pages 469 - 483; Ferguson, ACLST, Chapter 16-17, pages 107-118.
Due: Monday, January 11, 2010.

1. Suppose that $(X,Y),(X_1,Y_1),\ldots,(X_n,Y_n)$ are i.i.d. with bivariate normal distribution $N_2(\mu,\Sigma)$ where $\mu \in R^2$ and

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma\tau\rho \\ \sigma\tau\rho & \tau^2 \end{pmatrix}$$

   where $\sigma^2 > 0$, $\tau^2 > 0$, and $\rho \in (-1,1)$.
   (a) If we assume that $\mu_1 = \mu_2 \equiv \theta$ and Σ is known, what is the MLE of θ?
   (b) If we assume that μ is known and $\tau^2 = c^2\sigma^2 \equiv c^2\theta$ with $c > 0$ fixed and known, what is the MLE of $\theta = \sigma^2$?
   (c) What is the asymptotic distribution of the estimator you found in (b)?
   (d) Under the same assumption as in (b), what is the MLE of ρ?
   (e) What is the asymptotic distribution of the estimator you found in (d)?

2. Ferguson, ACILST, page 118, problem 3. [Also see Lehmann and Casella, Example 7.9, page 482.]

3. Ferguson, ACILST, page 117, problem 2, with parameter space $\Theta = [0,1]$.

4. Ferguson, ACILST, page 124, problem 3. What can you say about the asymptotic distribution of the MLE of $\theta = (\theta_1, \theta_2)$?

5. (Profile likelihood) [For nice plots to accompany this exercise, see pages 41 - 43 of Cox, D. R. and Oakes, D. (1984); *Analysis of Survival Data*, Chapman and Hall.] As in problem 1.3, consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{[0,\infty)}(x)$$

   where $\theta \equiv (\alpha,\beta) \in (0,\infty)\times(0,\infty) \subset \mathbb{R}^2$.
   (a) For a sample of n observations from p_θ, we know that, for each fixed value of β the value of α which maximizes the likelihood as a function of α is

$$\widehat{\alpha}(\beta) = \{\frac{1}{n}\sum_{i=1}^n X_i^\beta\}^{1/\beta}.$$

   Use this to compute the *profile likelihood* $l_{\text{profile}}(\beta) = l_{\text{profile}}(\beta|\underline{X})$ defined by

$$l_{\text{profile}}(\beta) = l(\widehat{\alpha}(\beta),\beta) = l(\widehat{\alpha}(\beta),\beta|\underline{X}).$$

   (b) Use what we know from Statistics 581 problem 10.3 to show that the profile likelihood is strictly concave and hence has a unique maximum. Show that maximizing the profile likelihood as a function of β yields the maximum likelihood estimate: i.e. that $(\widehat{\alpha}, \widehat{\beta}) = (\widehat{\alpha}(\widehat{\beta}_{\text{profile}}), \widehat{\beta}_{\text{profile}})$.

6. **Optional bonus problem 1.** This is a continuation of problem 5 above.
   (a) What is the relationship of the score function for β from the profile likelihood, $\dot{l}_{\beta,\text{profile}}$ to the (efficient) score for β from the full likelihood? Prove or disprove my claim: the profile score for β (based on n observations) is asymptotically equivalent to the sum of efficient scores for β over the sample in the sense that their difference divided by $\sqrt{n}$ converges to 0 in probability.
   (b) What is the relationship of the observed information from the profile likelihood $-\ddot{l}_{\beta\beta,\text{profile}}$ to information quantities from the full likelihood?

7. **Optional bonus problem 2.** Ferguson, ACILST, page 125, problem 7.