Statistics 582, Final Exam

Wellner; 3/16/2009; 4/10/09 (ver2)

Instructions: This is an "in class" and "closed-book" exam. Please do it completely on your own with no books or notes.

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A uniformly most powerful level α test.

   (b) An unbiased test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (c) A similar on the boundary test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A level α permutation test.

   (e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.

2. (30 points) **State** *any three* of the following results:

   (a) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.

   (b) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   (c) A theorem about admissibility properties of the sample mean $\overline{X}$ when sampling from a normal distribution on $\mathbb{R}$ and a contrasting theorem for sampling from a normal distribution on $\mathbb{R}^d$.

   (d) The generalized Neyman - Pearson lemma (in the "short form" stated in the notes).

   (e) A conditional limit theorem about the large sample behavior of posterior distributions.

   Do either problem 3 or problem 4.

3. (36 points) **State** and **prove** the Neyman - Pearson lemma.

4. (36 points) A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.1	.05	.35	.5
$p_1(x)$	.3	.25	.35	.1

(a) For the usual $0-1$ loss, find a most powerful test of size .10 for testing $H: p = p_0$ versus $K: p = p_1$ and determine its power.

(b) Find a test ϕ which minimizes the sum of risks $E_0\phi + E_1(1-\phi)$.

(c) If the losses are $L(1,1) = L(0,0) = 0$, $L(0,1) = 10$, $L(1,0) = 5$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.7, .3)$, find the Bayes rule d_B and the minimax rule d_M.

5. (48 points) Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. Consider testing $H: \ p_2 \le p_1$ versus $K: \ p_2 > p_1$.

(a) Write the joint density (probability mass function) $p(x, y; p_1, p_2) \equiv P_{p_1,p_2}(X = x, Y = y)$ in exponential family form $c(\theta, \xi)\exp(\theta U(x, y) + \xi T(x, y))h(x, y)$ where $T(x, y)$ is sufficient for the boundary $\Theta_B = \{(p_1, p_2) \in [0, 1]^2 : \ p_1 = p_2\}$ and $U(x, y) = y$.

(b) Show carefully that testing H versus K is equivalent to testing $H_1: \theta \le 0$ (ξ = anything) versus $K_1: \theta > 0$ (ξ = anything).

(c) What is the conditional distribution of $U(X, Y) = Y$ given $T = t$ under $p_1 = p_2$? Compute it explicitly and give its name.

(d) Find the UMP unbiased test of size α as explicitly as possible when $m = 3$, $n = 2$, $t = 2$, and $\alpha = 1/10$.

(e) Relate the conditional distribution of the test statistic involved in (c) to a problem involving sampling without replacement from a finite population. Identify the contents of the urn (i.e. the numbers on the balls in the urn) and calculate the mean and variance of Y given $T = t$ in this conditional distribution.

(f) Use the results of (e) together with the Wald-Wolfowitz-Noether-Hájek CLT to show a conditional (on $T = t$) CLT for Y appropriately centered and normalized if $0 < \liminf(m/N) \le \limsup(m/N) < 1$. [Make sure that you verify the key hypothesis of the theorem.]

Do either problem 6 or problem 7.

6. (36 points) Suppose that $X_i \sim \text{exponential}(\theta/i)$, $i = 1, \ldots, n$are independent; thus the density $p_i(\cdot; \theta)$ of X_i is given by

$$p_i(x_i; \theta) = \frac{\theta}{i}\exp(-\theta x_i/i)1_{[0,\infty)}(x_i). \tag{1}$$

A. Is there a UMP test of $H: \theta \le \theta_0$ versus $K: \theta > \theta_0$? If so, find it and specify the critical point as precisely as possible.
B. What is the MLE of θ?

C. Now suppose that (3) gives the conditional distribution of the X_i's given θ and that $\theta \sim \Gamma(\alpha, \beta)$ Find the Bayes test of H versus K for $0-1$ loss; express this as precisely as possible in terms of a rejection set involving the X_i's. Compare this rejection set to that of the UMP test.

7. (36 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. exponential(θ) random variables so that $1 - F_\theta(x) = \exp(-\theta x)$ for $x \geq 0$ and $\theta > 0$.
   (a) Find the MLE of θ.
   (b) If $\theta \sim \text{Gamma}(\alpha, \beta)$ so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta)$$

   and $E(\theta) = \alpha/\beta$, find the posterior distribution of θ.
   (c) Find the Bayes estimator of θ for squared error loss. Is it consistent?
   (d) What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?