Statistics 582, Midterm Exam

Wellner; 2/13/2009

1. (24 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.
(a) An *admissible* decision rule.
(b) A *Bayes rule* with respect to a prior distribution Λ.
(c) A *minimax decision rule*.
(d) A *least favorable prior distribution*.
(e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.
(f) The *Kullback-Leibler information* $K(P, Q)$ between two probability distributions P and Q on a measurable space $(\mathcal{X}, \mathcal{A})$.

2. (24 points) **State** *any two* of the following results:
(a) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.
(b) Any theorem / result about nonparametric maximum likelihood estimation.
(d) A uniform strong law of large numbers (or Glivenko - Cantelli theorem).
(e) Wald's theorem on strong consistency of maximum likelihood estimators.

Do **either** problem 3 **or** problem 4.

3. (32 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. Uniform$(0, \theta)$ conditional on $\boldsymbol{\theta} = \theta$, and suppose that $\boldsymbol{\theta}$ has the Pareto (θ_0, α) prior with density

$$\lambda(\theta) = \left(\frac{\alpha}{\theta_0}\right)\left(\frac{\theta_0}{\theta}\right)^{\alpha+1} 1_{(\theta_0,\infty)}(\theta),$$

with prior mean

$$E(\boldsymbol{\theta}) = \frac{\alpha}{\alpha - 1}\theta_0 \qquad \text{if } \alpha > 1.$$

(a) Find the posterior distribution of $\boldsymbol{\theta}$.
(b) Find the Bayes estimator of $\boldsymbol{\theta}$ for squared-error loss.
(c) For what values of θ is the Bayes estimator consistent?
(d) Compute the risk $R(\theta, d)$ of the usual unbiased estimator of θ, $d_{ub}(\underline{X}) = (n + 1)X_{(n)}/n$.

4. (32 points) Suppose that $(X|\boldsymbol{\theta}) \sim$ Poisson$(\boldsymbol{\theta})$; i.e.

$$p(x|\theta) = e^{-\theta}\frac{\theta^x}{x!}, \qquad x \in \{0, 1, 2, \ldots\},$$

and the prior distribution of $\boldsymbol{\theta}$ is Gamma(α, β), i.e.

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta).$$

(a) Find the posterior distribution of $\boldsymbol{\theta}$.
(b) Find the Bayes estimator of θ for squared error loss, $L(\theta, a) = (\theta - a)^2$.
(c) Find the Bayes estimator for testing $H_0: \ \theta \in (0, 3]$ versus $H_1: \ \theta \in (3, \infty)$.
(d) Find the Bayes estimator of θ for the loss function $L(\theta, a) = (\theta - a)^2/\theta$.

Do **either** problem 5 **or** problem 6.

5. (30 points) Suppose that $\Theta = \{0, 1\} = \mathcal{A}$ with losses $L(0, 0) = L(1, 1) = 0$, $L(0, 1) = L(1, 0) = 1$. Suppose that P_0 and P_1 have densities p_0 and p_1 with respect to a dominating measure μ, and suppose that P_0 and P_1 have prior probabilities $1 - \lambda$ and λ respectively for some $\lambda \in [0, 1]$.
(a) Show that the Bayes rules d_Λ (with $d_\Lambda(x)$ =probability of action 1 given that $X = x$ is observed) have the form

$$d_\Lambda(x) = \begin{cases} 1 & \text{if } \ \lambda p_1(x) > (1-\lambda) p_0(x), \\ \gamma(x) & \text{if } \ \lambda p_1(x) = (1-\lambda) p_0(x), \\ 0 & \text{if } \ \lambda p_1(x) < (1-\lambda) p_0(x). \end{cases}$$

(b) Give an expression for the Bayes risk for the prior λ.
(c) Show that when $\lambda = 1/2$ the Bayes risk is related to the total variation distance between P_0 and P_1.

6. (30 points) Consider testing $H_0: \ X \sim U(0, 1)$ versus $H_1 : X \sim U(1/2, 3/2)$ with zero - one loss.
(a) Find the risk set $\mathcal{R}$.
(b) Find the minimax rule.
(c) Find the least favorable prior distribution Λ.

7. (30 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. with mixture density (mass function)

$$p(x; \lambda, \mu, \theta) = \theta \frac{\lambda^x}{x!} e^{-\lambda} + (1 - \theta) \frac{\mu^x}{x!} e^{-\mu}, x = 0, 1, \ldots,$$

where $0 < \theta < 1$, $0 < \lambda \neq \mu < \infty$; in other words, p is the mixture of two Poisson distributions with parameters λ and μ respectively.
(a) Describe an EM - algorithm for estimation of (λ, μ, θ).
(b) What is the natural corresponding nonparametric model for the data which were modelled with the parametric mixture distribution in (a)? What is the natural nonparametric maximum likelihood estimator for the nonparametric model?