Statistics 582, Problem Set 6 Solutions

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1. Consider Example 5.5.4 on pages 16 and 17 of the Chapter 5 notes.
(a) Show that the variance of $\hat{\psi}$ is given by

$$Var(\hat{\psi}_n) = \frac{1}{n}\left\{\frac{1}{B}\sum_{j=1}^{B}\frac{\theta_j}{\xi_j} - \psi(\theta)^2\right\}.$$

[Hint: use the formula $Var(Y) = EVar(Y|X) + Var[E(Y|X)]$ twice.]
(b) Use the result of (a) to show that

$$Var(\hat{\psi}_n) \le \frac{1}{n\delta}$$

under the assumption that $\xi_j \ge \delta > 0$ for all $1 \le j \le B$.

Solution: (a) Since the (X_i, R_i, Y_i)'s are i.i.d.,

$$\begin{aligned}
Var(\hat{\psi}_n) &= n^{-1}Var\left(\frac{R_1Y_1}{\xi_{X_1}}\right) \\
&= n^{-1}\left\{EVar\left(\frac{R_1Y_1}{\xi_{X_1}}|R_1, X_1\right) + Var\left(E\left(\frac{R_1Y_1}{\xi_{X_1}}|R_1, X_1\right)\right)\right\} \\
&= n^{-1}\left\{E\left(\frac{R_1^2}{\xi_{X_1}^2}\theta_{X_1}(1-\theta_{X_1})\right) + Var\left(\frac{R_1}{\xi_{X_1}}\theta_{X_1}\right)\right\} \\
&= n^{-1}\left\{EE\left(\frac{R_1^2}{\xi_{X_1}^2}\theta_{X_1}(1-\theta_{X_1})|X_1\right)\right. \\
&\qquad \left. + EVar\left(\frac{R_1}{\xi_{X_1}}\theta_{X_1}|X_1\right) + Var\left(E\left(\frac{R_1}{\xi_{X_1}}\theta_{X_1}|X_1\right)\right)\right\} \\
&= n^{-1}\left\{E\left(\frac{\theta_{X_1}(1-\theta_{X_1})}{\xi_{X_1}}\right)\right. \\
&\qquad \left. + E\left(\frac{\theta_{X_1}^2}{\xi_{X_1}^2}\xi_{X_1}(1-\xi_{X_1})\right) + Var(\theta_{X_1})\right\} \\
&= n^{-1}\left\{\frac{1}{B}\sum_{j=1}^{B}\frac{\theta_j(1-\theta_j)}{\xi_j} + \frac{1}{B}\sum_{j=1}^{B}\theta_j^2\frac{1-\xi_j}{\xi_j} + \frac{1}{B}\sum_{j=1}^{B}(\theta_j - \overline{\theta})^2\right\} \\
&= n^{-1}\left\{\frac{1}{B}\sum_{j=1}^{B}\frac{\theta_j}{\xi_j} - \psi(\theta)^2\right\}.
\end{aligned}$$

(b) Since $\xi_j \geq \delta$ and $\theta_j \leq 1$ for all j, it follows that

$$Var(\hat{\psi}_n) \leq n^{-1} \frac{1}{B} \sum_{j=1}^{B} \frac{1}{\delta} = \frac{1}{n\delta}.$$