Statistics 582, Final Exam Solutions

Wellner; 3/17/2009

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A uniformly most powerful level α test.

   (b) An unbiased test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (c) A similar on the boundary test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A level α permutation test.

   (e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.

   Solution: See class notes.

2. (30 points) **State** *any three* of the following results:

   (a) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.

   (b) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   (c) A theorem about admissibility properties of the sample mean $\overline{X}$ when sampling from a normal distribution on $\mathbb{R}$ and a contrasting theorem for sampling from a normal distribution on $\mathbb{R}^d$.

   (d) The generalized Neyman - Pearson lemma (in the "short form" stated in the course notes).

   (e) A conditional limit theorem about the large sample behavior of posterior distributions.

   Solution: See class notes.

   Do either problem 3 or problem 4.

3. (36 points) **State** and **prove** the Neyman - Pearson lemma.

   Solution: See class notes.

4. (36 points) A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.06	.08	.38	.48
$p_1(x)$	.18	.32	.19	.31

(a) For the usual $0-1$ loss, find a most powerful test of size .10 for testing $H: p = p_0$ versus $K: p = p_1$ and determine its power.

(b) Find a test ϕ which minimizes the sum of risks $E_0\phi + E_1(1-\phi)$. What is the relationship between the minimized sum of risks and the total variation distance between P_1 and P_2 and why does this make intuitive sense?

(c) If the losses are $L(1,1) = L(0,0) = 0$, $L(0,1) = 3$, $L(1,0) = 4$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.6, .4)$, find the Bayes rule d_B and the minimax rule d_M.

Solution:
(a). The ratios of the probabilities under the two hypotheses are given by the following table

x	1	2	3	4
$p_0(x)$	.06	.08	.38	.48
$p_1(x)$	.18	.32	.19	.31
$p_1(x)/p_0(x)$	3	4	1/2	31/48

Thus by the Neyman - Pearson lemma the most powerful test of size .10 is $\phi(x) = 1\{x = 2\} + (1/3)1\{x = 1\}$. The power of this test is $\beta_\phi = E_1\phi(X) = P_1(X = 2) + (1/3)P_1(X = 1) = .32 + (1/3)(.18) = .32 + .06 = .38$.
(b). The sum of risks $E_0\phi + E_1(1-\phi) = 1 + \sum_x \phi(x)\{p_0(x) - p_1(x)\}$ is minimized by any rule $\phi(x) = 1\{x : p_1(x) > p_0(x)\} = 1_{\{1,2\}}(x)$. The minimum sum of risks is $E_0\phi(X) = 1 + (p_0(1) - p_1(1)) + (p_0(2) - p_1(2)) = 1 - .12 - .24 = .64$. [Note that the total risk of the Neyman-Pearson test of size .10 is $.1 + .62 = .72$.] The total variation distance between P_0 and P_1 is $d_{TV}(P_0, P_1) = 1 - \eta(P_0, P_1)$ where $\eta(P_0, P_1) = \int p_0 \wedge p_1 d\mu = .06 + .08 + .19 + .31 = .64$. Thus

$$\min_\phi\{E_0\phi + E_1(1-\phi)\} = \eta(P_0, P_1) = 1 - d_{TV}(P_0, P_1)$$

This is intuitively reasonable: if $d_{TV}(P_0, P_1)$ is large, then we can make the minimum sum of errors small, but if $d_{TV}(P_0, P_1)$ is small, then the minimum sum of errors will be large.
(c). For a rule $d = (d_1, d_2, d_3, d_4)$ which chooses 1 with probability d_x when x is

observed, the (ordinary) risks are

$$\begin{aligned}
R(0,d) &= E_0 L(0,d) = 3\{d_1(.06) + d_2(.08) + d_3(.38) + d_4(.48) \\
&= \{.18d_1 + .24d_2 + 1.14d_3 + 1.44d_4\}, \\
R(1,d) &= E_1 L(1,d) \\
&= 2\{(1-d_1)(.18) + (1-d_2)(.32) + (1-d_3)(.19) + (1-d_4)(.31)\} \\
&= 2 - .36d_1 - .64d_2 - .38d_3 - .62d_4.
\end{aligned}$$

Thus the Bayes risk for the prior $\lambda = (.4, .6)$ is

$$\begin{aligned}
\mathcal{R}(\lambda, d) &= .4R(0,d) + .6R(1,d) \\
&= \frac{1}{10}\{.72d_1 + .96d_2 + 4.56d_3 + 5.76d_4\} \\
&\qquad + \frac{1}{10}\{12 - (2.16d_1 + 3.84d_2 + 2.28d_3 + 2.48d_4)\} \\
&= \frac{1}{10}\{12 + (.72 - 2.16)d_1 + (.96 - 3.84)d_2 \\
&\qquad + (4.56 - 2.28)d_3 + (5.76 - 2.48)d_4\}.
\end{aligned}$$

Since the coefficients of d_1 and d_2 are negative, and the coefficients of d_3 and d_4 are positive, the Bayes rule is given by $d_B = (1,1,0,0)$ with corresponding Bayes risk $\mathcal{R}(\lambda, d_B) = (12 - 1.44 - 2.88)/10 = 7.68/10 = .768$. The ordinary risks of this rule are $R(0, d_B) = .42$ and $R(1, d_B) = 1$. [Crosscheck: $.4R(0,d_B) + .6R(1,d_B) = .4(.42) + .6(1) = .768$.]

To find the minimax rule d_M, we first consider the non-random rule $d = (1,1,1,0)$. This rule has ordinary risks $R(0,d) = 1.86$, $R(1,d) = 1.62$. Thus we suspect that a minimax rule is of the form $d' = (1,1,0,d_4')$. Any rule of this form has risks $R(0,d') = .42 + 1.44d_4'$, $R(1,d') = 1 - .62d_4'$. Equating these and solving for d_4' yields $2.06d_4' = 0.58$, or $d_4' = 58/206 = 29/103$. Then $R(0,d') = .42 + 1.44(29/103) = .825437$, while $R(1,d') = 1 - .62(29/103) = .825437$. Thus the particular rule $d_M = (1,1,0,29/103)$ is minimax.

5. (42 points) (From problem set #8, modified.) Suppose that $X_1, \ldots, X_m$ are i.i.d. Exponential$(1/\mu)$ (with density $p(x;\mu) = \mu^{-1}\exp(-x/\mu)1_{(0,\infty)}(x)$) and suppose that $Y_1, \ldots, Y_n$ are i.i.d. Exponential$(1/\nu)$ (with density $p(y;\nu) = \nu^{-1}\exp(-y/\nu)1_{(0,\infty)}(y)$) and independent of the X_i's. Consider testing $H: \nu \le \mu$ versus $K: \nu > \mu$.
(a) Find a sufficient statistic $T = T(\underline{X}, \underline{Y})$ for $\theta \in \Theta_B = \overline{\Theta}_0 \cap \overline{\Theta}_1$.
(b) Find a UMP unbiased test of size $\alpha \in (0,1)$ of H versus K in the form of a conditional test.
(c) Show that the UMPU test you found in (b) can be carried out unconditionally by use of an appropriate ancillary statistic and Basu's theorem.

Solution:
(a) The joint density of the data is given by

$$\begin{aligned}
p_{\mu,\nu}(\underline{x},\underline{y}) &= \mu^{-m}\nu^{-n}\exp\left(-\frac{1}{\mu}\sum_{i=1}^{m}x_i - \frac{1}{\nu}\sum_{j=1}^{n}y_j\right) \\
&= \mu^{-m}\nu^{-n}\exp\left(\left(\frac{1}{\mu}-\frac{1}{\nu}\right)\sum_{i=1}^{n}y_j - \frac{1}{\mu}\left(\sum_{i=1}^{n}x_i + \sum_{j=1}^{n}y_j\right)\right) \\
&= \mu^{-m}\nu^{-n}\exp\big(\theta U(\underline{x},\underline{y}) + \xi T(\underline{x},\underline{y})\big)
\end{aligned}$$

where

$$\theta \equiv \frac{1}{\mu}-\frac{1}{\nu}, \qquad U(\underline{x},\underline{y}) = \sum_{j=1}^{n}y_j,$$

$$\xi \equiv -\frac{1}{\mu}, \qquad T(\underline{x},\underline{y}) = \sum_{i=1}^{n}x_i + \sum_{j=1}^{m}y_j.$$

It follows that on the boundary set $\Theta_B = \{(\mu,\mu) : \mu > 0\}$ the statistic $T = T(\underline{X},\underline{Y}) = \sum_i X_i + \sum_j Y_j$ is sufficient for $\xi = -1/\mu$.
(b) From the derivation in (a) we see that testing H versus K is equivalent to testing $\theta \le 0$ versus $\theta > 0$. Thus a UMP unbiased test of H versus K is given by

$$\phi(\underline{X},\underline{Y}) = \begin{cases} 1 & \text{if } \sum_1^n Y_j > c(T) \\ \gamma(T) & \text{if } \sum_{i=1}^n \sum_1^n Y_j = c(T) \\ 0 & \text{if } \sum_{j=1}^n Y_j < c(T). \end{cases}$$

where $c(T)$, $\gamma(T)$ are chosen so that $E_{(\mu,\mu)}(\phi(\underline{X},\underline{Y})|T) = \alpha$.
(c) Now on the boundary $\nu = \mu$, $V = U/T = \sum Y_j/(\sum X_i + \sum Y_j) \sim \text{Beta}(n,m)$ is ancillary, and hence independent of T by Basu's theorem. Since V is a monotone function of U for each fixed $T = t$, it follows that the test in (b) can be re-expressed as $\phi(\underline{X},\underline{Y}) = 1\{V > c\}$ where c is chosen so that $P_{\mu,\mu}(V > c) = \int_c^1 \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} v^{n-1}(1-v)^{m-1}dv = \alpha$. Alternatively, since $U \equiv \sum Y_j \sim \text{Gamma}(n,\mu^{-1})$ and $W \equiv \sum X_i \sim \text{Gamma}(m,\mu^{-1})$ on the boundary $\nu = \mu$, it follows that $2\mu^{-1}U \sim \chi^2_{2n}$ and $2\mu^{-1}W \sim \chi^2_{2m}$ are independent, so

$$\frac{U/(2n)}{W/(2m)} = \frac{m}{n}\frac{U}{W} \sim F_{2n,2m}$$

and, moreover

$$V = U/T = U/(U+W) = \frac{U/W}{1+U/W}$$

is a monotone increasing function of U/W, so the UMPU test can be carried out unconditionally using tables of the $F_{2n,2m}$ distributions.

6. (42 points) Consider testing

$$H_c: \; X_1, \ldots, X_m, Y_1, \ldots, Y_n \;\text{ are i.i.d. }\; F \in \mathcal{F}_c$$

versus

$$K_1: \; X_1, \ldots, X_m, Y_1, \ldots, Y_n \;\text{ have joint density }\; h$$

where h is given by

$$\begin{aligned} h(\underline{x}, \underline{y}, \mu, \nu) \;\; &= \;\; \mu^{-m} \exp\left(-\sum_1^m x_i/\mu\right) \prod_{i=1}^m 1_{(0,\infty)}(x_i) \\ &\quad \cdot \nu^{-n} \exp\left(-\sum_1^n y_j/\nu\right) \prod_{j=1}^n 1_{(0,\infty)}(y_j) \end{aligned}$$

where $\nu > \mu$ are both fixed.
(a) Find a MP similar test of H_c versus K_1; express your test in terms of either $\overline{Y}_n - \overline{X}_m$ or $\sum_{j=1}^n Y_j$.
(b) Does the resulting test in (a) depend on the particular $\nu > \mu$ which were specified? State the resulting property of the test derived in (a).
(c) Now consider comparing the test you derived in problem 4(c) to the similar test derived in (a) and (b). How would you show that the test in problem 4(c) is asymptotically equivalent to the test in (a) for large n in the sense that the critical points are approximately equal for large n? What conditions are needed?

Solution:
(a) To find a MP similar test of H_c we condition on the order statistics $\underline{Z}$ of the pooled sample, $\underline{Z} = (Z_1, \ldots, Z_N)$ with $N = m + n$. To find a most powerful similar test of H_c versus K_1, we reject H_c for those permutations $\underline{z}'$ of $\underline{Z}$ for which

$$\mu^{-m}\nu^{-n} \exp\left(-\frac{1}{\mu}\sum x_i - \frac{1}{\nu}\sum y_j\right)$$

is large; or, equivalently, if

$$\frac{1}{\mu}\sum x_i + \frac{1}{\nu}\sum y_j$$

is small; or equivalently if

$$\left(\frac{1}{\nu} - \frac{1}{\mu}\right)\sum y_j + \frac{1}{\mu}\left(\sum x_i + \sum y_j\right)$$

is small; or, equivalently (since $\nu > \mu$ for the alternative K_1), if

$$\sum_{j=1}^{n} y_j$$

is large. Thus we reject if $\sum_{j=1}^{n} Y_j > c_\alpha(\underline{Z})$ where $c_\alpha(\underline{Z})$ is chosen so that we reject for exactly $I = \alpha N!$ of the $N!$ permutations of $\underline{Z}$.

(b) The resulting test does not depend on which particular $\nu > \mu$ are used to specify K_1, so in fact the resulting test is UMP similar for testing H_c versus $K \equiv \cup_{\mu<\nu} K_{1,\mu,\nu}$.

(c) To try compare the two tests we try to rewrite them in the same form with different critical points, and then compare the critical points where the comparison will very likely involve an application of the WWNH CLT in the case of the permutation test.

We start by rewriting the test in 5(c): since $U/T \sim \text{Beta}(n,m)$, $E(U/T) = n/(n+m) = n/N$, and the test of 5(c) is equivalent to rejecting if

$$\frac{U}{T} - \frac{n}{N} > b_{n,m,\alpha} - \frac{n}{N},$$

or if

$$\frac{U}{T} \cdot \frac{N}{n} - 1 > b_{n,m,\alpha}\frac{N}{n} - 1,$$

where the left side can be rewritten, with $\lambda_N \equiv m/N$, $1 - \lambda_N = n/N$, as

$$\begin{aligned} \frac{\overline{Y}}{\lambda_N \overline{X} + (1-\lambda_N)\overline{Y}} - 1 &= \frac{\overline{Y} - (1-\lambda_N)\overline{Y} - \lambda_N \overline{X}}{\lambda_N \overline{X} + (1-\lambda_N)\overline{Y}} \\ &= \frac{\lambda_N}{\lambda_N \overline{X} + (1-\lambda_N)\overline{Y}}(\overline{Y} - \overline{X}). \end{aligned}$$

Hence the test is, equivalently, "reject if

$$\frac{\sqrt{\frac{mn}{N}}(\overline{Y} - \overline{X})\lambda_N}{\lambda_N \overline{X} + (1-\lambda_N)\overline{Y}} > \sqrt{\frac{mn}{N}}\left(b_{n,m,\alpha}\frac{N}{n} - 1\right),$$

or to

$$S_{m,n} \equiv \frac{\sqrt{\frac{mn}{N}}(\overline{Y} - \overline{X})}{\lambda_N \overline{X} + (1-\lambda_N)\overline{Y}} > \frac{1}{\lambda_n}\sqrt{\frac{mn}{N}}\left(b_{n,m,\alpha}\frac{N}{n} - 1\right). \tag{1}$$

Here, assuming that $\lambda_N \to \lambda \in (0,1)$ and (only) that the X's and Y's have a common d.f. F with $E_F X^2 < \infty$, it follows that

$$\begin{aligned}
\sqrt{\frac{mn}{N}}(\overline{Y}_n - \overline{X}_m) &= \sqrt{\frac{m}{N}}\sqrt{n}(\overline{Y} - \mu) - \sqrt{\frac{n}{N}}\sqrt{m}(\overline{X} - \mu) \\
&\to_d \sqrt{\lambda}N_1(0, \sigma^2) - \sqrt{1-\lambda}N_2(0, \sigma^2) \\
&\qquad \text{where the two normal rv's are independent} \\
&\sim N(0, \sigma^2),
\end{aligned}$$

and where $\lambda\overline{X} + (1 - \lambda_N)\overline{Y} \to_p \mu$. Hence it follows that

$$S_{m,n} \to_d \frac{1}{\mu}N(0, \sigma^2) = N(0, \sigma^2/\mu^2)$$

in general when $E_F X^2 < \infty$, and in the exponential special case this becomes $N(0, 1)$ (since $\sigma^2 = \mu^2$). This implies that the right side in (1) converges to z_α satisfying $P(Z > z_\alpha) = \alpha$.

For the permutation form of the test, note that (using the notation of section 6.2.x),

$$\begin{aligned}
S_{m,n} &= \frac{\sqrt{\frac{mn}{N}}(\overline{Y} - \overline{X})}{\overline{Z}} \\
&= \frac{\sqrt{\frac{mn}{N}}\frac{N}{n}(\overline{Y} - \overline{Z})}{\overline{Z}} \\
&= \frac{\overline{Y} - \overline{Z}}{\sigma_N} \cdot \frac{\sigma_N\sqrt{\frac{mn}{N}}}{\overline{Z}} \\
&= \frac{\overline{Y} - \overline{Z}}{\sigma_N} \cdot \frac{\sqrt{\frac{N}{N-1}}\sigma_z}{\overline{Z}} \\
&\to_d N(0, 1) \cdot \frac{\sigma}{\mu}.
\end{aligned}$$

Thus we see that the permutation test is *not* asymptotically equivalent to the parametric test of problem 5(c), but is in fact compensating for the fact that $\sigma^2 \neq \mu^2$ for a general F on $\mathbb{R}^+$.

Do either problem 7 or problem 8.

7. (36 points) Suppose that $X_i \sim N(\Delta i, 1)$, $i = 1, \ldots, n$ with $\Delta > 0$.
(a) Find a UMP size α test of $H : \Delta \le 0$ versus $K : \ \Delta > 0$.
(b) Compute the power function of the test you found in (a) as explicitly as possible.
(c) Find an alternative test of size α of H versus K based on $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$.

(d) Compute the power function of the test you found in (c) as explicitly as possible.
(e) Compare the power functions you found in (b) and (d). Which test do you prefer?

Solution: (a) The joint density of the X_i's is given by

$$\begin{aligned} p_\Delta(\underline{x}) &= (2\pi)^{-n/2}\exp\left(-\sum_1^n (x_i - \Delta i)^2/2\right) \\ &= (2\pi)^{-n/2}\exp\left(\Delta\sum_1^n ix_i - \frac{1}{2}\Delta^2\sum_1^n i^2\right)\exp(-\sum_1^n x_i^2/2) \end{aligned}$$

and this has MLR in $T(\underline{X}) = \sum_1^n iX_i$. Thus by the Karlin - Rubin theorem, the UMP test of H versus K is of the form $\phi(\underline{X}) = 1\{T(\underline{X}) > k\}$ where k is chosen so that $E_0\phi(\underline{X}) = P_0(T(\underline{X}) > k) = \alpha$. Now $T \sim N(ET, Var(T)) = N(\Delta\sum_1^n i^2, \sum_1^n i^2)$ where

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus $P_0(T > k) = P_0(N(0, n(n+1)(2n+1)/6) > k) = \alpha$ if $k = z_\alpha\sqrt{n(n+1)/(2n+1)/6}$.
(b) The power function of the UMP test derived in (a) is

$$\begin{aligned} \beta_\phi(\Delta) &= P_\Delta(T > z_\alpha\sqrt{n(n+1)(2n+1)/6}) \\ &= P_\Delta\left(\frac{T - \Delta n(n+1)(2n+1)/6}{\sqrt{n(n+1)(2n+1)/6}} > z_\alpha - \Delta\sqrt{n(n+1)(2n+1)/6}\right) \\ &= P(Z > z_\alpha - \Delta\sqrt{n(n+1)(2n+1)/6}). \end{aligned}$$

(c) For the alternative test based on $\overline{X}_n$ we have

$$E_\Delta(\overline{X}_n) = n^{-1}\sum_{i=1}^n i\Delta = (n+1)\Delta/2,$$
$$Var_\Delta(\overline{X}_n) = n^{-2}\sum_{i=1}^n 1 = n^{-1},$$

so we can use the test $\phi^*(\underline{X}) = 1\{\overline{X}_n > k_\alpha^*\}$ where k_α^* is determined by

$$\beta_{\phi^*}(0) = P_0(\overline{X}_n > k_\alpha^*) = P_0\left(\frac{\overline{X}_n - 0\cdot(n+1)/2}{\sqrt{1/n}} > \frac{k_\alpha^*}{\sqrt{1/n}}\right) = \alpha$$

and hence $k_\alpha^* = z_\alpha n^{-1/2}$.
(d) The power function ϕ^* of the test based on $\overline{X}_n$ is given by

$$\begin{aligned}
\beta_{\phi^*}(\Delta) &= P_\Delta(\overline{X}_n > z_\alpha n^{-1/2}) \\
&= P_\Delta\left(\frac{\overline{X}_n - \Delta(n+1)/2}{\sqrt{1/n}} > \frac{z_\alpha n^{-1/2} - \Delta(n+1)/2}{1/\sqrt{n}}\right) \\
&= P\left(Z > z_\alpha - \sqrt{n}(n+2)\Delta\right).
\end{aligned}$$

(e) Note that $\sqrt{n(n+1)(2n+1)/6} > \sqrt{n}(n+1)/2$ since $2n+1 > (3/2)(n+1)$ for $n > 1$ Hence we prefer the UMP test derived in (a) and (b) for $n > 1$. (When $n = 1$, the two tests coincide.)

Some further notes: In what might be more traditional and more general notation, the model in this problem would be written as $Y_i = \beta x_i + \epsilon_i$ for $1 \le i \le n$ where $\{x_1, \ldots, x_n\}$ are given numbers (in the problem as stated $x_i = i$, $1 \le i \le n$), and where the ϵ_i's are i.i.d. $N(0,1)$: I have replaced X_i by Y_i (which often used as the "response variable" in a regression setting), and I have change Δ to β, the usual notation for a slope. It follows that $Y_i \sim N(\beta x_i, 1)$ for $1 \le i \le n$ are independent, and hence the joint density of the Y_i's is given by

$$\begin{aligned}
p_\beta(\underline{y}) &= \prod_{i=1}^n (2\pi)^{-1/2} \exp\left(-\sum_{i=1}^n (y_i - \beta x_i)^2/2\right) \\
&= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\left(\sum_{i=1}^n y_i^2 - 2\beta \sum_{i=1}^n x_i y_i + \beta^2 \sum_{i=1}^n x_i^2\right)\right) \\
&= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^n y_i^2\right) \cdot \exp\left(\beta \sum_{i=1}^n x_i y_i\right) \cdot \exp\left(-\beta^2 \sum_1^n x_i^2/2\right)
\end{aligned}$$

which is an exponential family model with MLR in $T(\underline{Y}) = \sum_{i=1}^n x_i Y_i$. As above, the UMP test of $H: \beta \le 0$ versus $K: \beta > 0$ is given by $\phi(\underline{Y}) = 1\{T(\underline{Y}) > k\}$ where k is chosen so that $E_0\phi(\underline{Y}) = P_0(T(\underline{Y}) > k) = \alpha$. But since $T(\underline{Y}) = \sum_1^n x_i Y_i \sim N(\beta \sum_1^n x_i^2, \sum_{i=1}^n x_i^2)$, it follows that we will achieve the correct size α if $k = z_\alpha \sqrt{\sum_1^n x_i^2}$. The calculation of the power function is similar to the derivation above, so I will stop here.

8. (36 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. Cauchy$(\theta, 1)$; i.e. with common density $p(x;\theta) = \pi^{-1}(1 + (x-\theta)^2)^{-1}$, $\theta \in \mathbb{R}$. The locally most powerful test of $H: \theta \le 0$ versus $K: \theta > 0$ we derived in example 1.5 is "reject H if $S_n(0) > k_\alpha$" where

$$S_n(0) = n^{-1/2} \sum_{i=1}^n \frac{2X_i}{1 + X_i^2}$$

and where the choice $k_\alpha = 2^{-1/2} z_\alpha$ yields $P_0(S_n(0) > k_\alpha) \to \alpha$ as $n \to \infty$. As we saw in Example 1.5, this test ϕ has the property that

$$\beta_\phi(\theta) \equiv P_\theta(S_n(0) > 2^{-1/2} z_\alpha) \to 0 \qquad \text{as} \;\; \theta \to \infty$$

for each fixed n.
(a) Consider the test $\phi^*(\underline{X})$ given by $\phi^*(\underline{X}) = 1\{M_n > k_\alpha^*\}$ where $M_n \equiv \mathbb{F}_n^{-1}(1/2)$ is the median of the X_i's. How should k_α^* be chosen so that

$$\beta_{\phi^*}(0) = P_0(M_n > k_\alpha^*) \to \alpha \qquad \text{as} \;\; n \to \infty? \tag{2}$$

(b) Let $Y_1, \ldots, Y_n$ be i.i.d. Cauchy$(0,1)$ with density $p(x;0) = \pi^{-1}(1+x^2)^{-1}$. Show that $M_n(\underline{X}) \stackrel{d}{=} M_n(\underline{Y}) + \theta$.
(c) Use the result of (b) to compute $\beta_{\phi^*}(\theta) = P_\theta(M_n > k_\alpha^*)$. Use this to show that $\beta_{\phi^*}(\theta) \to 1$ as $\theta \to \infty$ for each fixed $n \geq 1$.

Solution:
(a) Recall that if $f(F^{-1}(1/2)) > 0$, then the sample median $\mathbb{F}_n^{-1}(1/2)$ satisfies

$$\sqrt{n}(\mathbb{F}_n^{-1}(1/2) - F^{-1}(1/2)) \to_d -Q'(1/2)\mathbb{U}(1/2) \sim N(0, \frac{1/4}{f^2(F^{-1}(1/2))}).$$

In the present problem, the distribution functions $F(x;\theta)$ are given by

$$F(x;\theta) \;\; = \;\; \int_{-\infty}^{x} p(y;\theta)dy = F_0(x-\theta)$$

where F_0 is the standard Cauchy distribution function

$$F_0(x) = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+y^2}dy = \frac{1}{2} + \frac{1}{\pi} arctan(x).$$

Thus

$$F^{-1}(u;\theta) = \theta + F_0^{-1}(u)$$

where $F_0^{-1}(u) = \tan(\pi(u - 1/2))$, and it follows that $F^{-1}(1/2,\theta) = \theta$, $f_\theta(F^{-1}(1/2,\theta)) = \pi^{-1}$. Hence

$$\sqrt{n}(\mathbb{F}_n^{-1}(1/2) - \theta) \to_d N(0, \pi^2/4).$$

Thus the convergence in (2) holds if $k_\alpha^* = n^{-1/2}(\pi/2)z_\alpha$ where $P(Z > z_\alpha) = \alpha$.
(b) If $Y_1, \ldots, Y_n$ are i.i.d. Cauchy$(0,1)$, then $X_i \equiv Y_i + \theta$, $i = 1, \ldots, n$ are i.i.d. Cauchy$(\theta, 1)$, and hence $(X_{(1)}, \ldots, X_{(n)}) \stackrel{d}{=} (Y_{(1)}, \ldots, Y_{(n)}) + \theta\underline{1}$, and this implies that $M_n(\underline{X}) \stackrel{d}{=} M_n(\underline{Y}) + \theta$.

(c) From (b) we find that the power of the test based on the sample median M_n is given by

$$\begin{aligned}
\beta_{\phi^*}(\theta) &= P_\theta(M_n(\underline{X}) > k_\alpha^*) \\
&= P(M_n(\underline{Y}) + \theta > n^{-1/2}(\pi/2)z_\alpha) \\
&= P(M_n(\underline{Y}) > n^{-1/2}(\pi/2)z_\alpha - \theta) \to 1
\end{aligned}$$

monotonically for each fixed n as $\theta \to \infty$. Thus the test based on M_n does not suffer from the same biasedness difficulty as does the locally most powerful test.

It should also be noted (although this was not part of the problem) that the local (asymptotic) power of the test based on $S_n(0)$ does exceed the local asymptotic power of the test based on the median: if $\theta_n \equiv t/\sqrt{n}$, then

$$\begin{aligned}
P_{\theta_n}(S_n(0) > 2^{-1/2}z_\alpha) &= P_{\theta_n}(S_n(0) - E_{\theta_n}S_n(0) + E_{\theta_n}S_n(0) > 2^{-1/2}z_\alpha) \\
&\to P(2^{-1/2}Z + t/2 > 2^{-1/2}z_\alpha) = P(Z + 2^{-1/2}t > z_\alpha)
\end{aligned}$$

since, with $Y_i \equiv 2X_i/(1+X_i^2) = \dot{\mathbf{l}}_\theta(X_i;0)$,

$$\begin{aligned}
E_{\theta_n}S_n(0) - E_0S_n(0) &= \sqrt{n}\,\{E_{\theta_n}Y_1 - E_0Y_1\} \\
&\to tE_0Y_1 \cdot \dot{l}_\theta(X_1;0) = tI(\theta) = t/2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P_{\theta_n}(M_n > (\pi/2)z_\alpha n^{-1/2}) &= P(\sqrt{n}(M_n - \theta_n) + \sqrt{n}\theta_n > (\pi/2)z_\alpha) \\
&\to P((\pi/2)Z + t > (\pi/2)z_\alpha) = P(Z + (2/\pi)t > z_\alpha).
\end{aligned}$$

Here we have $.62662... = 2/\pi < 1/\sqrt{2} = .707107...$, so the local asymptotic power of the LMP test based on $S_n(0)$ is indeed locally (asymptotically) most powerful.