Statistics 582, Problem Set 7

Wellner; 2/18/2009

Reading: Chapter 6, sections 6.1-6.3;
Due: Wednesday, February 25, 2009.

1. Continuation of problem 3, problem set 4: Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have distribution P_θ and density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on $\mathbb{R}$, and that $\theta \sim \Gamma(\alpha, \beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

In problem set 4 we found the Bayes rules with respect to squared error loss $L(\theta, a) = (\theta - a)^2$ and weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$.

(a) Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
(b) What does theorem 5.8.2 say about the limiting distribution of the Bayes rule for squared error loss (assuming that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \equiv P$ with $\theta_0 \in (0, \infty)$)?

2. A random variable X takes on the values $1,\ 2,\ 3,\ 4$ with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.18	.06	.36	.40
$p_1(x)$	.36	.18	.24	.22

A. Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
B. Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$.

3. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \ P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

4. (Problem 3.7, Lehmann and Romano, TSH, page 94.) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	θ	2θ	$.9 - 2\theta$	$.1 - \theta$

where $0 < \theta < .1$. For testing $H: \theta = .05$ against $\theta > .05$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = .5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
(a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H: \theta = \theta_0 = 0$ versus $K: \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
(b) What happens to the risk body as n grows or as $\sigma_0 \to 0$?
(c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
(d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : \ n \geq 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?

6. **Optional bonus problem:** Suppose that $X_i \sim N(i\Delta, 1)$, $i = 1, \ldots, n$ are independent. Show that there exists a UMP test of $H: \Delta \leq 0$ versus $K: \Delta > 0$, and determine it as explicitly as possible.