Statistics 582, Problem Set 5

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Reading: Chapter 5, sections 6-8
Due: Wednesday, February 11, 2009.
Reminder: Midterm Exam: Friday, February 13.

1. Let $\mathcal{X} = \{0, 1\}$, $\mathcal{A} = \Theta = \{1, 2\}$, and assume that the losses are given by $L(1, 1) = L(2, 2) = 0$, $L(1, 2) = a$, $L(2, 1) = b$. Suppose that the statistician can observe either X or Y where

$$p_1(1) = P_1(X = 1) = 2/3, \qquad p_2(1) = P_2(X = 1) = 1/2,$$
$$p_1^*(1) = P_1(Y = 1) = 3/4, \qquad p_2^*(1) = P_2(Y = 1) = 1/2.$$

   Let $\underline{\lambda} = (\lambda, 1 - \lambda)$, $\lambda \in [0, 1]$ be the prior distribution over Θ.
   (a) Find the Bayes risk when X is observed, and similarly for Y.
   (b) In the case $a = b$, $\lambda = 1/2$, would the statistician prefer to observe X or Y?
   (c) For general $a \neq b$, $\lambda \in (0, 1)$ would the statistician prefer to observe X or Y?

2. Let $\Theta = \mathcal{A} = \{0, 1\}$, assume the (hypothesis testing) loss function $L(0, 0) = L(1, 1) = 0$, $L(0, 1) = L(1, 0) = 1$. Suppose that we observe the random variable X with the discrete distribution $P_\theta(X = x) = 2^{-(x+\theta)} 1_{\mathbb{Z}^+ \cap [1-\theta, \infty)}(x)$. (a) Describe the set of all non-randomized decision rules.
   (b) Plot the risk set $\mathcal{R}$ in the plane. Which non-randomized rules are admissible? Why?
   (c) Can you find a non-randomized minimax rule?
   (d) What decision rules result from a Neyman-Pearson approach?
   Hint: every number $z \in [0, 1]$ has a diadic representation of the form $\sum_{x=1}^{\infty} d(x) 2^{-x}$ where $d(x) = 0$ or 1.

3. Suppose that $X_n \equiv X \sim \text{Multinomial}_k(n, \underline{\theta})$.
   (a) Suppose that the prior distribution on θ is given by a Dirichlet distribution, Dirichlet($\underline{\alpha}$):

$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} 1_{[\underline{\theta}:\ \sum \theta_i = 1]} \,.$$

   Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4
   (b) What is the posterior distribution for θ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
   (c) Find a minimax estimator d_M of $\underline{\theta}$.

4. Find the limit distribution of the minimax estimator d_M in problem 3 (i.e. $\sqrt{n}(d_M(X_n) - p) \to_d$ "something" and find "something"). Is d_M a regular estimator of p?

5. Let $\Theta = (0, \infty)$, $\mathbf{A} = [0, \infty)$, let X have the discrete distribution

$$p(x, \theta) = \binom{r + x - 1}{x} \theta^x (\theta + 1)^{-(r+x)}, \qquad x = 0, 1, 2, \ldots$$

where r is some known positive integer; this is the negative binomial distribution reparametrized so that $E_\theta X = r\theta$. Suppose that

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta(\theta + 1)}.$$

(a) Show that the usual estimator, $d_0(X) = X/r$ is an equalizer rule; i.e. show that it has a risk function $R(\theta, d_0)$ which is constant in θ.
(b) Show that the usual estimator d_0 is generalized Bayes with respect to Lebesgue measure on $(0, \infty)$ provided $r > 1$. (A generalized Bayes rule is a rule that minimizes the posterior Bayes risk even when starting with an improper prior; see e.g. Ferguson, MS, page 50.) (What happens if $r = 1$?)
(c) Find Bayes decision rules with respect to the prior distributions $\Lambda_{\alpha,\beta}$ with densities

$$\lambda_{\alpha,\beta}(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha - 1} (\theta + 1)^{-(\alpha+\beta)} 1_{(0,\infty)}(\theta),$$

the distribution of $\theta = Z/(1 - Z)$ where $Z \sim$ Beta(α, β).
(d) Show that $d(X) = X/(r + 1)$ is minimax. [Note that d_0 is not minimax, hence not admissible.]

6. **Optional bonus problem:** (Compare with Lehmann and Casella, TPE, Examples 5.1 and 5.2, pages 254-255.)
(a) Let $(X|\sigma^2) \sim N(0, \sigma^2)$. Show that the conjugate prior for σ^2 is the distribution of $1/Y$ where Y has a gamma distribution.
(b) Suppose that $(X|\theta, \kappa) \sim N(\theta, 1/\kappa)$, $(\theta|\kappa) \sim N(\mu, \tau/\kappa)$, and $\kappa \sim$ Gamma(α, β). Show that the posterior distribution of (θ, κ) has the same form as the prior.
(c) Find the marginal posterior distribution for θ in (b).
(d) If $X_1, \ldots, X_n$ are i.i.d. as X in (b), find the limiting distribution of the Bayes estimator of θ for squared-error loss.