Statistics 582, Problem Set 8 Solutions

Wellner; 3/7/2007

1. (a) Show that the logistic distribution with location parameter θ having density

$$p_\theta(x) = \frac{\exp(x-\theta)}{(1+\exp(x-\theta))^2} = \frac{1}{2(1+\cosh(x-\theta))}$$

has monotone likelihood ratio.

(b) Unfortunately the result of (a) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ as in (a), then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

Solution: (a) Let $\theta' > \theta$. Then the ratio of densities is given by

$$\begin{aligned} \frac{p_{\theta'}(x)}{p_\theta(x)} &= \frac{e^{x-\theta'}}{(1+e^{x-\theta'})^2} \cdot \frac{(1+e^{x-\theta})^2}{e^{x-\theta}} \\ &= e^{\theta-\theta'} \left(\frac{1+e^{x-\theta}}{1+e^{x-\theta'}} \right)^2 . \end{aligned}$$

This is a monotone increasing function of x if and only if its logarithm is a monotone increasing function of x. The logarithm is given by

$$\begin{aligned} \log\left(\frac{p_{\theta'}(x)}{p_\theta(x)}\right) &\equiv R(x;\theta,\theta') \equiv R(x) \\ &= \theta - \theta' + 2\log\left(\frac{1+e^{x-\theta}}{1+e^{x-\theta'}}\right) \\ &= \theta - \theta' + 2\left\{\log(1+e^{x-\theta}) - \log(1+e^{x-\theta'})\right\}, \end{aligned}$$

where R has derivative (with respect to x) R' given by

$$\begin{aligned} R'(x) &= 2\left\{\frac{e^{x-\theta}}{1+e^{x-\theta}} - \frac{e^{x-\theta'}}{1+e^{x-\theta'}}\right\} \\ &= \frac{2e^x}{(1+e^{x-\theta})(1+e^{x-\theta'})} \cdot (e^{-\theta} - e^{-\theta'}) \\ &> 0. \end{aligned}$$

Thus the family $\{p_\theta\}$ has monotone likelihood ratio in $T(x) = x$.
(b) As in Example 6.1.5, the locally best test is the one-sided score test, reject if $S_n(\theta_0) > k$ where

$$S_n(\theta_0) \;=\; \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{\mathbf{l}}_\theta(X_i;\theta_0).$$

Straightforward calculation yields

$$\dot{\mathbf{l}}_\theta(x) = 2\left\{\frac{1}{1+e^{-(x-\theta)}} - 1\right\} = 2\left\{F(x-\theta) - 1\right\}$$

where $F(x) = 1/(1+e^{-x})$ is the standard logistic distribution function. Thus for $\theta_0 = 0$ the test statistic is

$$S_n(0) = n^{-1/2} \sum_{i=1}^{n} 2\left\{\frac{1}{1+e^{-X_i}} - 1\right\},$$

and we reject for large values of $S_n(0)$. Since $S_n(0) \to_d N(0,1/3)$ under $\theta_0 = 0$, taking the constant k to be $3^{-1/2} z_\alpha$ leads to approximate size α for large n.

2. Continuation of problem 1, problem set 7:
(a) For P_0 and P_1 as given in problem 1 of problem set # 7, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0,P_1) = \int \sqrt{p_0 p_1}\, d\mu$.
(b) For the product laws P_{0n} and P_{1n} (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. P_0 or P_1 respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20,\ 50,\ 100$.
(c) What does this imply about the test, ϕ_n say, based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

Solution: First,

$$d_{TV}(P_0, P_1) = (1/2)\{.18 + .12 + .12 + .18\} = (1/2)(.6) = .3\,.$$

Furthermore,

$$\rho(P_0, P_1) = \sqrt{(.18)(.36)} + \sqrt{(.06)(.18)} + \sqrt{(.36)(.24)} + \sqrt{(.40)(.22)} = 0.949068$$

so that $H^2(P_0,P_1) = 1 - \rho(P_0,P_1) = .0509318...$, and $H(P_0,P_1) = 0.225681...$ Note that the inequalities of problem xxx are indeed satisfied:

$$H^2(P_0,P_1) \le d_{TV}(P_0,P_1) \le H(P_0,P_1)(1+\rho(P_0,P_1))^{1/2} \le \sqrt{2} H(P_0,P_1),$$

which in this case becomes:

$$.0509318 < .3 < 0.225681(1 + .0509318...)^{1/2} = .315071... < .0.319161....$$

For $n = 20,\ 50,\ 100$ we have

n	$\rho(P_0^n, P_1^n)$	$H(P_0^n, P_1^n)$	$H^2(P_0^n, P_1^n)$
1	0.949068	0.225681	0.0509318
20	0.351519	0.805283	0.648481
50	0.073260	0.962673	0.926739
100	0.00536713	0.997313	0.994633

Since the test $\phi = \phi(\underline{X})$ which minimizes the sum of risks has

$$\begin{aligned} E_0\phi(\underline{X}) + E_1(1-\phi(\underline{X})) &= \int p_0(\underline{x}) \wedge p_1(\underline{x}) d\mu(\underline{x}) \\ &\le \rho(P_0^n, P_1^n) \\ &= \rho^n(P_0, P_1) \to 0. \end{aligned}$$

From the table above we see that this happens quite rapidly.

3. Consider the Locally Most Powerful test ϕ for testing $H : \theta \le 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.
(a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.
(b) What is the behavior of each of these two approximations for large values of θ? Which of them shows that the power function decreases to 0 as $\theta \to \infty$? Why?

Solution: (a) The test is "reject H if $\sqrt{n}\,\overline{Y}_n > 2^{-1/2} z_\alpha$" where $Y_i \equiv 2X_i/(1+X_i^2)$ are i.i.d. and $X_i \sim$ Cauchy $(\theta, 1)$. Thus under P_θ, by using contour integration and Cauchy's formula, or by using Mathematica, Maple, or your favorite symbolic manipulation program,

$$\begin{aligned} m(\theta) \equiv E_\theta Y_i &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2} p_\theta(x) dx = \int_{-\infty}^{\infty} \frac{2x}{1+x^2} \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} dx \\ &= \frac{2\theta}{4+\theta^2}, \end{aligned}$$

and

$$\begin{aligned} \sigma^2(\theta) &\equiv Var_\theta(Y_i) = E_\theta Y_i^2 - m^2(\theta) \\ &= \frac{2(4+3\theta^2)}{(4+\theta^2)^2} - \left(\frac{2\theta}{4+\theta^2}\right)^2. \end{aligned}$$

For local alternatives $\theta = \theta_n = t/\sqrt{n}$, we have

$$\begin{aligned} \text{Power}(\theta_n) &= P_{\theta_n}(\sqrt{n}\,\overline{Y}_n > 2^{-1/2} z_\alpha) \\ &= P_{\theta_n}(\sqrt{n}(\overline{Y}_n - m(\theta_n)) \ge 2^{-1/2} z_\alpha - \sqrt{n}(m(\theta_n) - m(0))) \\ &\to P(2^{-1/2} Z \ge 2^{-1/2} z_\alpha - m'(0)t) \end{aligned}$$

where

$$
\begin{aligned}
m'(0) &= \int_{-\infty}^{\infty} \frac{2x}{1+x^2} \frac{d}{d\theta} p_\theta(x)|_{\theta=0} dx \\
&= \int_{-\infty}^{\infty} \dot{l}_\theta(x;0)\dot{l}_\theta(x;0) p_\theta(x;0) dx = I(\theta) = 1/2 \,.
\end{aligned}
$$

Hence we have

$$
\text{Power}(\theta_n) \to P(Z > z_\alpha - 2^{-1/2}t) = 1 - \Phi(z_\alpha - 2^{-1/2}t) \,.
$$

This approximation to the power function increases monotonically from α at $t = 0$ to 1 at $t = \infty$ (effectively when $t > 2^{1/2} \cdot 4$). Note that this result is very much in qualitative agreement with corollary 4.2.4 from Statistics 581.

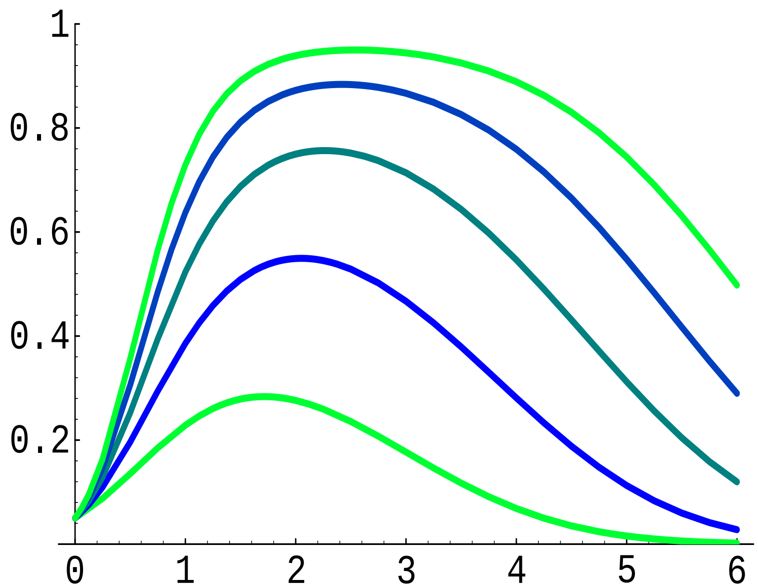

Figure 1: Plots of fixed θ power approximations for $n = 3, 6, 9, 12, 15$.

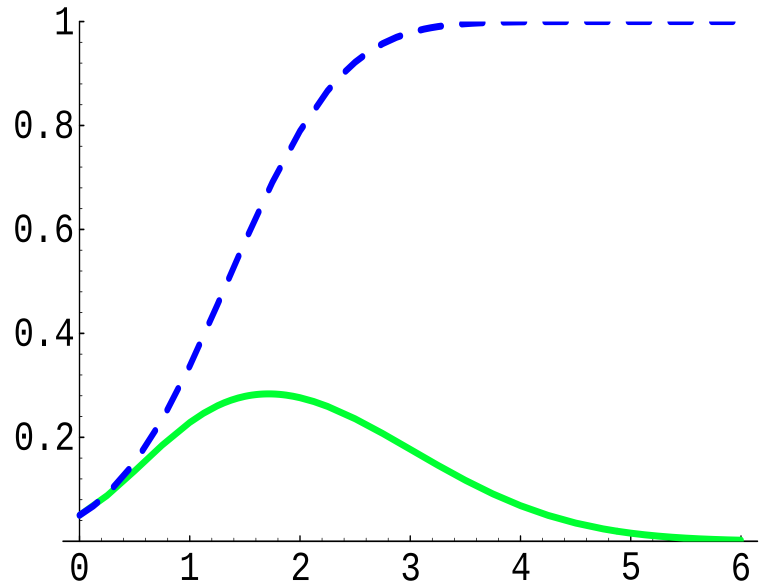

Figure 2: Plot of local and fixed θ power approximations, $n = 3$.

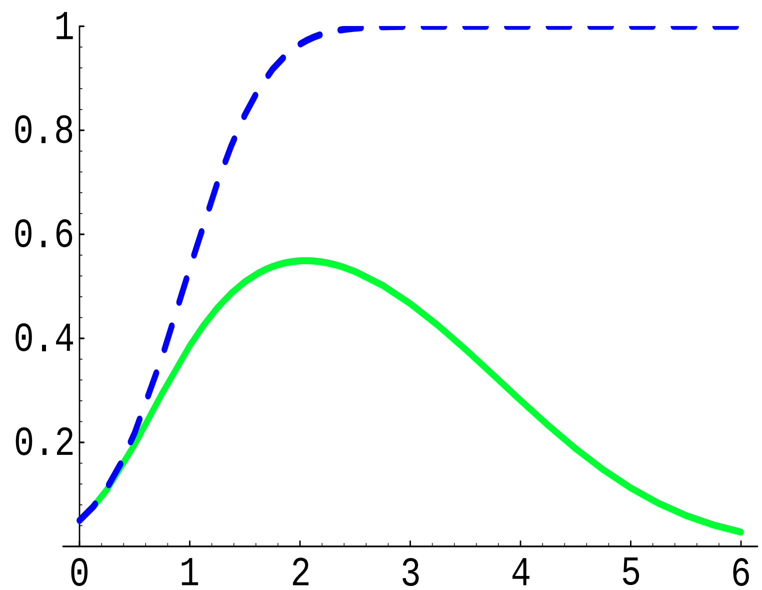

Figure 3: Plot of local and fixed θ power approximations, $n = 6$.

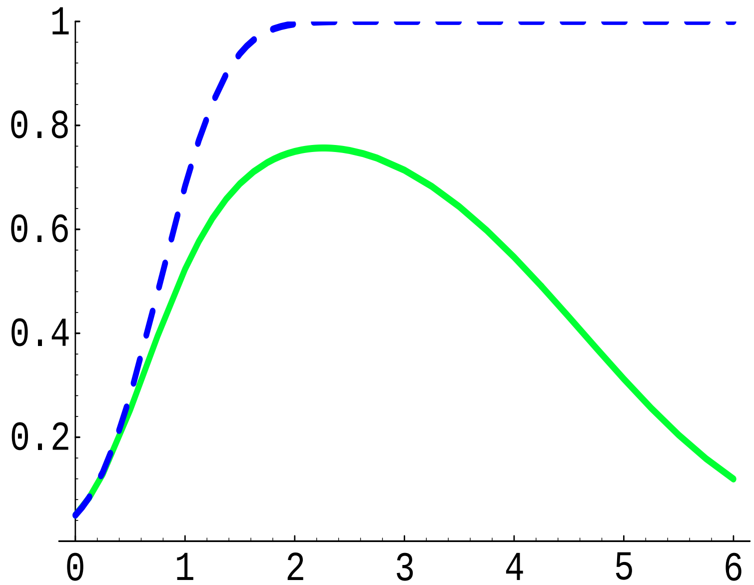

Figure 4: Plot of local and fixed θ power approximations, $n = 9$.

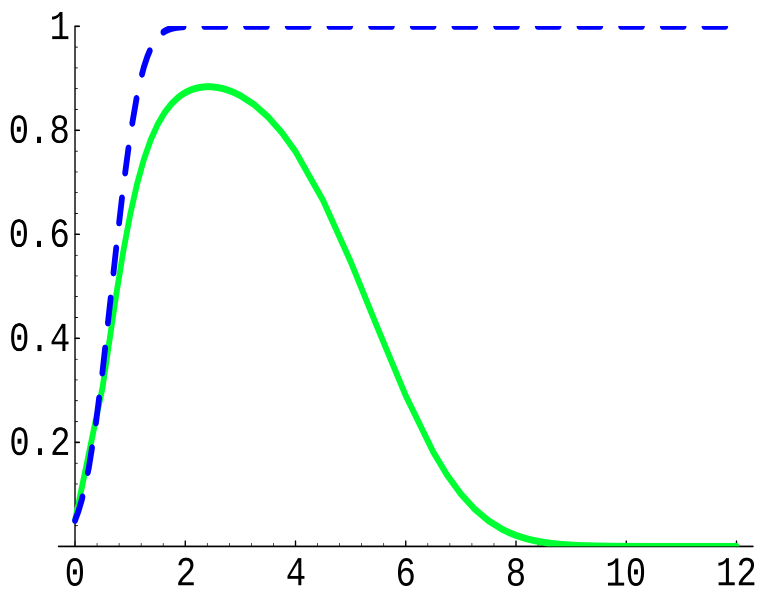

Figure 5: Plot of local and fixed θ power approximations, $n = 12$.

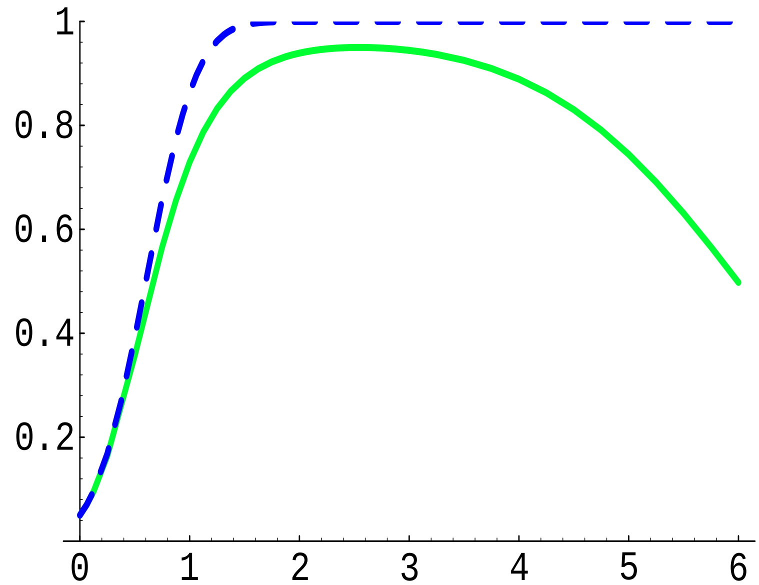

Figure 6: Plot of local and fixed θ power approximations, $n = 15$.

(b) For fixed alternatives $\theta > 0$ we have

$$\begin{aligned}
\text{Power}(\theta) &= P_\theta(\sqrt{n}\,\overline{Y} > 2^{-1/2} z_\alpha) \\
&= P_\theta(\sqrt{n}(\overline{Y}_n - m(\theta)) > 2^{-1/2} z_\alpha - \sqrt{n} m(\theta)) \\
&\doteq P(Z > (2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta)) \\
&= 1 - \Phi((2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta))\,.
\end{aligned}$$

This approximation to the power function is completely determined by the behavior of the functions $m(\theta)$ and $\sigma(\theta)$. Inspection of the function $(2^{-1/2} z_\alpha - \sqrt{n} m(\theta))/\sigma(\theta)$ shows that it first decreases (as it should if the power is to increase), but then it reaches a minimum and increases thereafter, and hence this approximation to the power decreases to zero just as we argued that it must in class. See the attached plot for $\alpha = .05$ and $n = 3, 6, 9, 12, 15$. the two different approximations for various sample sizes. Note that the two approximations agree for θ's close to 0, but the local approximation is always monotone increasing, while the approximations with θ fixed show the approximate power decreasing to 0 as $\theta \to \infty$ as we know it must. [Also note that we still have not computed the exact (finite n) power functions; it would be interesting to know how close our approximations are to the "exact" power. I'm betting that they are reasonably good.]

4. Let $X_1, \ldots, X_n$ be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.

(a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.
(b) Find the subclass of the tests that are independent of θ_1. These are UMP tests of H_0 versus $H_1' : \theta > \theta_0$.
(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is UMP of size α for testing $H_0' : \theta \le \theta_0$ versus $H_1' : \theta > \theta_0$ but that ϕ is not admissible.
(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \le b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \ne \theta_0$.

Solution: (a) For testing $\theta = \theta_0$ versus $\theta = \theta_1 > \theta_0$, the class of all NP tests is given by tests of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) > k\theta_0^{-n} 1_{[0,\theta_0]}(t) \\ \gamma(t), & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) = k\theta_0^{-n} 1_{[0,\theta_0]}(t) \\ 0, & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) < k\theta_0^{-n} 1_{[0,\theta_0]}(t) \end{cases} .$$

Thus for $k = (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t), & \text{if else ;} \end{cases}$$

for $k > (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t), & \text{if else} \\ 0, & \text{if } 0 \le t \le \theta_0; \end{cases}$$

and for $k < (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } 0 \le t \le \theta_1 \\ \gamma(t), & \text{if else.} \end{cases}$$

(b) The subclass of these tests that do not depend on θ_1 is the class of tests ϕ with

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t < \infty \\ \gamma(t), & \text{if else} \end{cases}$$

where $E_{\theta_0}\gamma(T) = \alpha$.
(c) The test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is of the form of the tests in (b) with $\gamma(t) = \alpha$ for $0 \le t \le \theta_0$, and hence is UMP of size α for testing $\theta = \theta_0$ versus $\theta > \theta_0$. To see that it is UMP for testing $\theta \le \theta_0$ versus $\theta > \theta_0$, we first compute its power function to confirm that it is of size α for the composite null hypothesis

$\theta \le \theta_0$. The power is

$$\begin{aligned}
\beta_\phi(\theta) &= E_\theta \phi(T) = P_\theta(T > \theta_0) + \alpha P_\theta(T \le \theta_0) \\
&= \{1 - (\frac{\theta_0}{\theta})^n\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta) + \alpha \left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0,\infty)}(\theta) \\
&= \{1 - (\frac{\theta_0}{\theta})^n (1-\alpha)\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta).
\end{aligned}$$

Thus we see that $\beta_\phi(\theta) = \alpha$ for $\theta \le \theta_0$, and ϕ is of size α for $\theta \le \theta_0$. Since the class of size α tests for testing $\theta = \theta_0$ is a larger class than the class of size α tests for testing $\theta \le \theta_0$, and since ϕ is UMP in the larger class, it is also UMP in the smaller class. Hence ϕ is UMP for testing $\theta \le \theta_0$ versus $\theta > \theta_0$. But the competing test $\phi_0(t) = 1\{t > (1-\alpha)^{1/n}\theta_0\}$ has power function

$$\begin{aligned}
\beta_{\phi_0}(\theta) &= E_\theta \phi_0(T) = P_\theta(T > \theta_0(1-\alpha)^{1/n}) \\
&= 1 - \{(1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0(1-\alpha)^{1/n},\infty)}(\theta) + 1_{[0,\theta_0(1-\alpha)^{1/n}]}(\theta)\},
\end{aligned}$$

so the power function of the test ϕ_0 is strictly below that of the test ϕ on the set $[0, \theta_0)$. Hence ϕ is inadmissible and the test ϕ_0 is also UMP.
(d) The test $\phi(t) = 1 - 1_{(\theta_0\alpha^{1/n},\theta_0]}(t)$ is of size α for testing $\theta = \theta_0$ versus $\theta \ne \theta_0$ since

$$E_{\theta_0}\phi(T) = P_{\theta_0}(T \le \theta_0\alpha^{1/n}) = \left(\frac{\theta_0\alpha^{1/n}}{\theta_0}\right)^n = \alpha.$$

Furthermore it is of the form of the class of all UMP tests for testing $\theta = \theta_0$ versus $\theta > \theta_0$, hence it UMP among the size α tests for $\theta > \theta_0$. For testing $\theta = \theta_0$ versus $\theta = \theta_1 < \theta_0$, the NP Pearson tests of the form $\phi_1(t) = \gamma(t)1_{[0,\theta_0]}(t)$ are most powerful of their size. But the test ϕ is of this form (with $\gamma(t) = 1_{[0,\theta_0\alpha^{1/n}]}(t)$), is in this class and does not depend on $\theta_1 < \theta_0$. Hence ϕ is UMP for testing $\theta = \theta_0$ versus $\theta \ne \theta_0$. [This is an unusual situation in which we get "something for free" from the structure of the uniform distributions. Usually two-sided tests are *not* UMP!]

5. Let X and Y be random variables with joint density

$$p_{X,Y}(x,y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y).$$

(a) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \le \mu + 1$ versus $H_1 : \lambda > \mu + 1$.
(b) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda = \mu$ versus

$H_1 : \lambda \neq \mu$.
(c) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \geq 2\mu$ versus $H_1 : \lambda < 2\mu$.
(d) What happens when $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ)?

Solution: When $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$ we have

$$p_{\lambda,\mu}(x, y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) .$$

(a) For testing $H : \lambda \leq \mu + 1$ versus $K : \lambda > \mu + 1$ we rewrite the density as follows:

$$\begin{aligned} p_{\lambda,\mu}(x, y) &= \lambda\mu \exp(-\lambda x - \mu y) \\ &= \lambda\mu \exp((\lambda - \mu)y - \lambda(x + y)) \\ &= \lambda\mu \exp(\theta U(x, y) + \xi T(x, y)) \end{aligned}$$

where $\theta \equiv \lambda - \mu$, $U(x, y) \equiv y$, $\xi \equiv -\lambda$, and $T(x, y) \equiv x + y$. Since $\lambda \leq \mu + 1$ is equivalent to $\lambda - \mu = \theta \leq 1 \equiv \theta_0$, our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } \; Y > c_\alpha(T) \\ \gamma(T) & \text{if } \; Y = c_\alpha(T) \\ 0 & \text{if } \; Y < c_\alpha(T) \end{cases}$$

where c_α and $\gamma(\alpha)$ satisfy $E\{\phi(X, Y)|T\} = \alpha$. In this case, the conditional distribution of Y given $T = X + Y$ on the boundary $\Theta_B = \{(\lambda - 1, \lambda) : \lambda \geq 1\}$ is given by

$$f_{Y|T}(y|t) = \frac{e^y}{e^t - 1} 1_{[0,t]}(y) .$$

Therefore $1 - F_{Y|T}(y|t) = 1 - (e^y - 1)/(e^t - 1)$ and for $\alpha = .2$ the critical point for the conditional test is given by

$$c_\alpha(T) = \log(\exp(T) - (\exp(T) - 1)/5) = \log((4/5)\exp(T) + 1/5) , \qquad \gamma(T) = 0 .$$

(b) For testing $H : \lambda = \mu$ versus $K : \lambda \neq \mu$, the same rewrite of the density as in (a) works. Now we have $\lambda = \mu$ is equivalent to $\mu - \lambda = 0 \equiv \theta_0$, and $\lambda \neq \mu$ is equivalent to $\mu - \lambda \neq 0 \equiv \theta_0$, so our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } \; Y > c_2(T) \; \text{ or } \; Y < c_1(T) \\ \gamma_i(T) & \text{if } \; Y = c_1(T) \; \text{ or } \; Y = c_2(T) \\ 0 & \text{if } \; c_1(T) < Y < c_2(T) \end{cases}$$

where the c_1, c_2, γ_1 and γ_2 are determined so that $E\{\phi(X,Y)|T\} = \alpha$. In this case the conditional distribution of Y given T on $\Theta_B = \{(\lambda, \lambda) : \lambda \geq 0\}$ is Uniform$(0, T)$, and hence the conditional distribution of Y/T given T is Uniform$(0, 1)$, and this is independent of T. Hence the UMPU test of H versus K of size .2 is given by "reject H if $Y/T < .1$ or $Y/T > .9$".
(c) For testing $H : \lambda \geq 2\mu$ versus $K : \lambda < 2\mu$, we need a somewhat different rewrite of the joint density. Now

$$\begin{aligned} p_{\lambda,\mu}(x,y) &= \lambda\mu \exp(-\lambda x - \mu y) \\ &= \lambda\mu \exp(-(\lambda - 2\mu)x - \mu(2x + y)) \\ &= \lambda\mu \exp(\theta U(x,y) + \xi T(x,y)) \end{aligned}$$

where $\theta \equiv 2\mu - \lambda$, $U(x,y) \equiv x$, $\xi \equiv -\mu$, and $T(x,y) \equiv 2x + y$. Since $\lambda \geq 2\mu$ is equivalent to $2\mu - \lambda \equiv \theta \leq 0 \equiv \theta_0$, (and $\lambda < 2\mu$ is equivalent to $2\mu - \lambda = \theta > 0 \equiv \theta_0$), our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if} \quad X > c_\alpha(T) \\ \gamma(T) & \text{if} \quad X = c_\alpha(T) \\ 0 & \text{if} \quad X < c_\alpha(T) \end{cases}$$

where $c_(T)$ and $\gamma(T)$ satisfy $E\{\phi(X,Y)|T\} = \alpha$. In this case the conditional distribution of X given T is Uniform$(0, T/2)$, so $2X/T$ is Uniform$(0, 1)$ and independent of T. Hence the UMPU test of H versus K of size $\alpha = .2$ is given by "reject H if $2X/T > .8$".

When we observe $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ), then the distribution of the observations is given by

$$\begin{aligned} p_{\lambda,\mu}(\underline{x}, \underline{y}) &= \lambda^m \mu^n \exp(-\lambda \sum_{i=1}^{m} x_i - \mu \sum_{j=1}^{n} y_j) \\ &= \lambda^m \mu^n \exp((\lambda - \mu) \sum_{j=1}^{n} y_j - \lambda(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)) \\ &= \lambda^m \mu^n \exp(\theta U(\underline{x}, \underline{y}) + \xi T(\underline{x}, \underline{y})) \end{aligned}$$

where $\theta \equiv \lambda - \mu$, $U(\underline{x}, \underline{y}) \equiv \sum y_j$, $\xi \equiv -\lambda$, and $T(\underline{x}, \underline{y}) \equiv \sum x_i + \sum y_j)$. This rewrite works for (a) and (b), and a similar rewrite works for (c) with the new $U = \sum x_i$, $T = 2\sum x_i + \sum y_j$. The form of the tests in (a) - (c) remains the same with the new U and T, and all that remains is to calculate the conditional distributions of U given T. In (a) this density is given by

$$f_{U|T}(u|t) = \frac{u^{n-1}(t-u)^{m-1}e^u}{\int_0^t v^{n-1}(t-v)^{m-1}e^v dv} .$$

In (b) it is easily found that $V \equiv U/T \sim \text{Beta}(n, m)$, and the test can be carried out unconditionally using tables of the Beta distributions. In (c) $2U \equiv \sum 2X_i \sim \text{Gamma}(m, \mu)$ and $V \equiv \sum Y_j \sim \text{Gamma}(n, \mu)$ on the boundary $\lambda = 2\mu$, so $2\mu U \sim \chi^2_{2m}$ and $\mu V \sim \chi^2_{2n}$. Therefore

$$\frac{2U/(2m)}{V/(2n)} = \frac{n}{m} 2\frac{U}{V} \sim F_{2m,2n}$$

and since $2U/T = 2U/(2U+V) = (2U/V)(1+(2U/V))$ is a monotone increasing function of $2U/V$, the UMPU test can be carried out unconditionally using tables of the $F_{2m,2n}$ distributions.