Statistics 582, Problem Set 7 Solutions

Wellner; 2/28/2007

1. A random variable X takes on the values $1,\ 2,\ 3,\ 4$ with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.18	.06	.36	.40
$p_1(x)$	.36	.18	.24	.22

A. Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
B. Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$.

Solution: A. Now $p_1(x)/p_0(x) = 2,\ 3,\ 2/3,\ 22/40 = 11/20$, according as $x = 1,\ 2,\ 3,\ 4$, so a MP test of size $\alpha = .2$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 2 \\ .14/.18, & \text{if } x = 1 \\ 0, & \text{if } x = 3, 4. \end{cases}$$

Then

$$E_0\phi(X) = P_0(X = 2) + \frac{.14}{.18}P_0(X = 1) = .06 + \frac{.14}{.18}.18 = .2,$$

while

$$\text{Power} = E_1\phi(X) = P_1(X = 2) + \frac{.14}{.18}P_1(X = 1) = .18 + \frac{.14}{.18}.36 = .18 + .28 = .46.$$

B. A test which minimizes $a + b$ is given by

$$\begin{aligned} \phi(x) &= \begin{cases} 1, & \text{if } p_1(x) \ge p_0(x) \\ 0, & \text{if } p_1(x) < p_0(x) \end{cases} \\ &= \begin{cases} 1, & \text{if } x = 1,\ 2 \\ 0, & \text{if } x = 3, 4 \end{cases} \end{aligned}$$

Then $a = E_0\phi(X) = .24$ and $b = E_1(1 - \phi(X)) = ..46$, and hence $(a + b)_{\min} = .7$. Note that $\int p_0 \wedge p_1 d\mu = .7$, so this result agrees with the solution of problem 4 of problem set # 4. Also note that for the NP test with $\alpha = .2$, the sum of the two types of error is $.2 + .54 = .74 > .7$.

2. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \ P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

Solution: Here the table of likelihood ratios is as follows:

x	1	2	3	4	5	6
$p_1(x)/p_0(x)$	2	5/2	4	2	∞	78/98
$p_2(x)/p_0(x)$	3	5/2	6	0	∞	72/98

(i) For $\alpha = .01$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1\{x = 5\} + (1/2)1\{x = 3\},$$
$$\phi_2(x) = 1\{x = 5\} + 1/2)1\{x = 3\},$$

so $\phi_1 = \phi_2$ is Uniformly most powerful.
(ii) For $\alpha = .05$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,3,5\}},$$

so there is no UMP test of P_0 versus P_1 and P_2 at this level.
(iii) For $\alpha = .07$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,2,3,5\}},$$

so by taking $\gamma(x) = 1\{x = 1\}$, $\phi_1(x) = \phi_2(x)$, and this test is Uniformly Most Powerful for testing P_0 versus P_1 and P_2.

3. (Problem 3.7, Lehmann and Romano, TSH, page 94.) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	θ	2θ	$.9-2\theta$	$.1-\theta$

where $0 < \theta < .1$. For testing $H : \theta = .05$ against $\theta > .05$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = .5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

Solution: The likelihood ratios $P_{\theta'}(X = x)/P_\theta(X = x)$

x	0	1	2	3
$P_\theta(X = x)$	θ	2θ	$.9 - .2\theta$	$.1-\theta$
$\frac{P_{\theta'}(X=x)}{P_\theta(X=x)}$	$\frac{\theta'}{\theta}$	$\frac{\theta'}{\theta}$	$\frac{9-20\theta'}{9-20\theta}$	$\frac{1-10\theta}{1-10\theta'}$

It is easy to check that

$$\frac{\theta'}{\theta} > \frac{9-20\theta'}{9-20\theta} > \frac{1-10\theta'}{1-10\theta}$$

Hence this family has monotone decreasing likelihood ratio in x (though not strictly), and strictly decreasing likelihood ratio in

$$\begin{aligned} T(x) &= 1\{x = 0\} + 1\{x = 1\} + 2 \cdot 1\{x = 2\} + 3 \cdot 1\{x = 3\} \\ &= x1\{x > 0\} + 1\{x = 0\}. \end{aligned}$$

It follows from the Karlin - Rubin theorem that a UMP test of $H : \theta \le \theta_0 = .05$ (of its level) is given by

$$\phi(X) = 1_{[T(X)<k]} + \gamma(X)1_{[T(X)=k]}. \tag{1}$$

(i) Note that the test $\phi_1(X) = 1\{X = 0\}$ is of the form (1) with $k = 1$ and $\gamma(X) = 1\{X = 0\}$ and it has level $\alpha = .05$; hence it is a UMP test of H versus K. The power of ϕ_1 is given by $\beta_1(\theta) \equiv E_\theta\phi_1(X) = \theta$.
(ii) The test $\phi_2(X) = .51\{X = 1\}$ is also of the form (1) with $k = 1$ and $\gamma(X) = .5 \cdot 1\{X = 1\}$ and it has level $\alpha = .05$. Hence it is also a UMP test of H versus K. The power of ϕ_2 is given by $\beta_2(\theta) \equiv E_\theta\phi_2(X) = \theta$.
(iii) The test $\phi_3(X) = 1\{X = 3\}$ is clearly not of the form (1). It has power function $\beta_3(\theta) = E_\theta\phi_3(X) = .1 - \theta$, so $\beta_3(.05) = .05$, but $\beta_3(\theta) > .05$ for $\theta < .05$ while $\beta_3(\theta) < .05$ for $\theta > \theta_0 = .05$. In fact, this is a UMP test of $\tilde{H} : \theta \ge \theta_0$ versus $\tilde{K} : \theta < \theta_0$.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
(a) Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
(b) What happens to the risk body as n grows or as $\sigma_0 \to 0$?
(c) What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
(d) What happens to the risk bodies $\{(R(\theta_0, \phi), R(\theta_{1,n}, \phi)) : \ n \geq 1\}$ when $\theta_1 \equiv \theta_{1,n} \equiv \theta_0 + cn^{-1/2}$?

Solution: A. If $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma_0)$, to find optimal tests ϕ we can reduce (by sufficiency) to consideration of $\overline{X} \sim N(\theta, \sigma_0^2/n)$. My favorite family of tests (in fact the most powerful tests) of H versus K are the tests $\phi_c(\underline{X}) = 1\{\overline{X} > c\}$.For these tests

$$\begin{aligned} R(0, \phi_c) &= E_0\phi_c(\underline{X}) = P_0(\overline{X} > c) \\ &= P_0(\sqrt{n}(\overline{X} - 0)/\sigma_0 > \sqrt{n}c/\sigma_0) \\ &= 1 - \Phi(\sqrt{n}c/\sigma_0) \end{aligned}$$

and

$$\begin{aligned} R(1, \phi_c) &= E_1(1 - \phi_c(\underline{X})) \\ &= P_1(\overline{X} \leq c) = P_1(\sqrt{n}(\overline{X} - 1) \leq \sqrt{n}(c - 1)) \\ &= \Phi(\sqrt{n}(c - 1)/\sigma_0). \end{aligned}$$

Since these tests are MP for testing H versus K, there are no points with risks below the curve given by $\{(R(0, \phi_c), R(1, \phi_c)) : \ c \in \mathbb{R}\}$; this is the lower boundary of the risk body. Note that the tests $\phi_{\text{ignore}}(\underline{X}) \equiv \alpha$ have risks $R(0, \phi_{\text{ignore}}) = \alpha$, $R(1, \phi_{\text{ignore}}) = 1 - \alpha$. Thus the line $\{(\alpha, 1 - \alpha) : \ \alpha \in [0, 1]\}$ is in the risk body. Furthermore, note that the tests $\phi_c'(\underline{X}) \equiv 1 - \phi_c(\underline{X}) = 1\{\overline{X} \leq c\}$ are MP for testing $H : \theta = 0$ versus $K' : \theta = \theta_1 < 0$, and by the Karlin - Rubin theorem these tests minimize the power function at points $\theta = \theta_1$ in the class of all tests with fixed power function (say at α) at $\theta = \theta_0$. Since

$$\text{Power}_{\phi_c'}(\theta) = E_\theta \phi_c' = 1 - R(\theta, \phi_c),$$

this says that the tests ϕ_c' maximize $R(1, \phi_c)$ over tests ϕ with $R(0, \phi) = \alpha$. Hence there are no points in the risk body with risks above the curve given by $\{(1 - R(0, \phi_c), 1 - R(1, \phi_c)) : \ c \in \mathbb{R}\}$.
B. As n grows or $\sigma_0 \to 0$ the risk body expands out toward the boundary of the square $[0, 1]^2$; see the plots below.
C. As $\theta_1 \to \theta_0 = 0$, the risk body contracts toward the diagonal line $(\alpha, 1 - \alpha)$ – since the testing problem becomes harder. See the plots below.

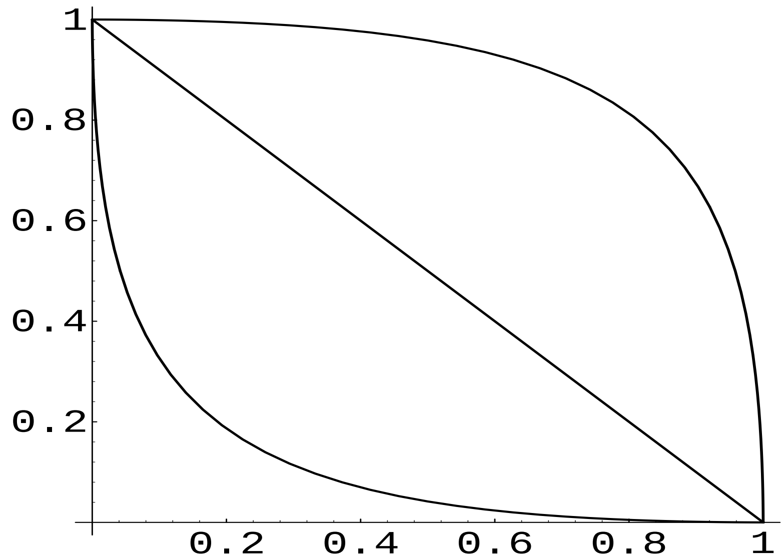

Figure 1: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = 1$

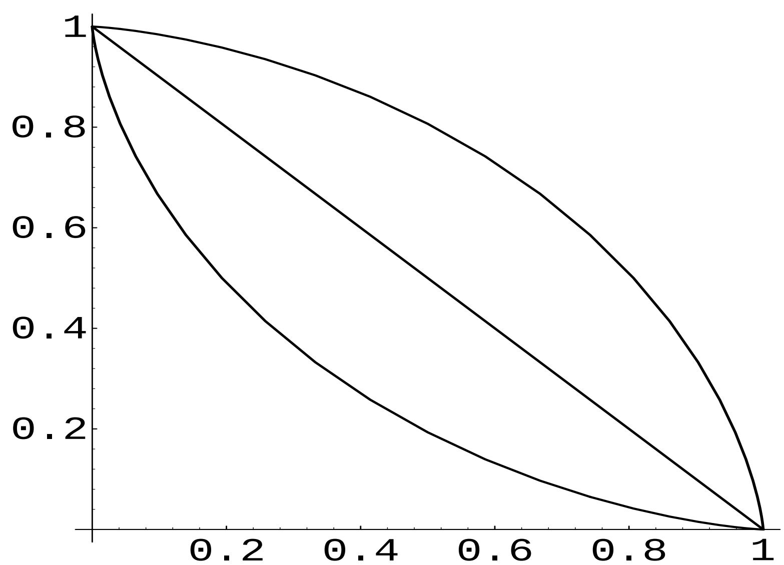

Figure 2: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = .5$

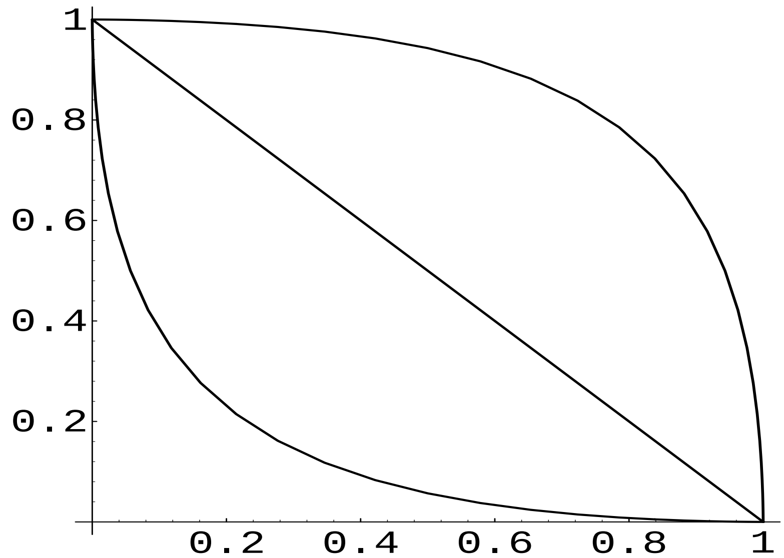

Figure 3: Risks for normal mean test, $n = 10$, $\sigma_0 = 1$, $\theta_1 = .5$

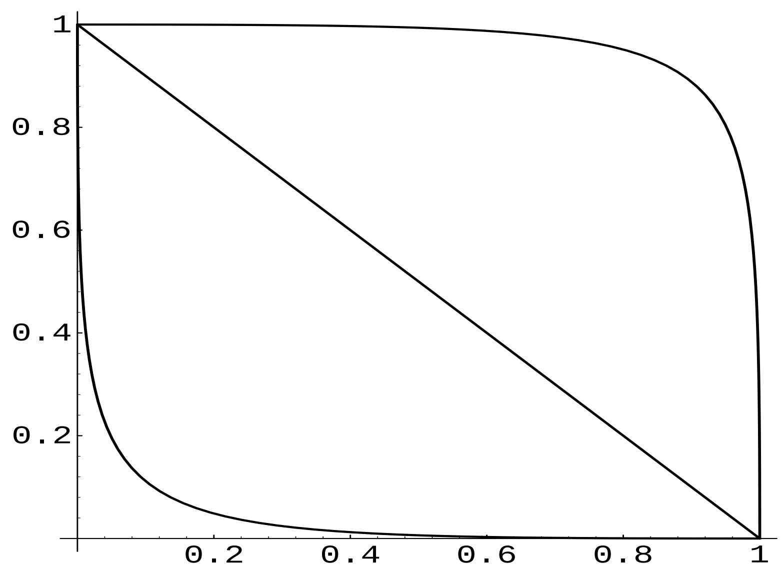

Figure 4: Risks for normal mean test, $n = 25$, $\sigma_0 = 1$, $\theta_1 = .5$