Statistics 582, Problem Set 6

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Due: Wednesday, February 15, 2006.
Reading: Chapter 5, section 8; start reading Chapter 6

1. Suppose that $X \sim P_\theta$ for $\theta \in \Theta \subset R^k$ has well-defined Fisher information matrix $I(\theta)$ for θ. The *Jeffreys prior* distribution Λ_J has density $\lambda_J(\theta) = \det(I(\theta))^{1/2}$ with respect to Lebesgue measure on Θ. Note that Λ_J may not be a finite measure, and even if Λ_J is a finite measure, it may not have total mass 1. If a prior distribution is a finite measure, then call it a *proper prior distribution*, and correspondingly if it is not a finite measure, call it an *improper prior distribution*. If the resulting posterior distribution is a finite measure, call it a *proper posterior distribution*, and (by convention) normalize it to have total mass 1. See Lehmann and Casella, TPE, pages 230, 234, 287, 305.
(a) Suppose that $X \sim \text{Bernoulli}(\theta)$. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.
(b) Suppose that $X \sim \text{Poisson}(\theta)$ with $\theta \in (0,\infty)$. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. Is it ever a proper posterior distribution?
(c) Suppose that $X \sim \text{Geometric}(\theta)$, i.e. the number of trials until the first success in i.i.d. Bernoulli trials with probability θ of success for each trial – recall Chapter 1, section 1. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,1))$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior. If we observe $X_1, \ldots, X_n$ i.i.d. Geometric(θ), so that $\sum X_i \sim$ Negative Binomial(n,θ) is the posterior distribution "proper" for some n?
(d) Suppose that $X \sim \text{Weibull}(\theta)$ with $\theta = (\alpha,\beta) \in (0,\infty)\times(0,\infty)$ as in chapters 3 and 4. Find the Jeffrey's prior density λ_J for θ. Is Λ_J a finite measure? If it is finite, what is $\Lambda_J((0,\infty)^2)$? Find the corresponding posterior distribution of Θ starting with the Jeffrey's prior.

2. Continuation of problem 3, problem set 4: Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have distribution P_θ and density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on $\mathbb{R}$, and that $\theta \sim \Gamma(\alpha,\beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

In problem set 4 we found the Bayes rules with respect to squared error loss $L(\theta, a) = (\theta - a)^2$ and weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$.

(a) Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
(b) What does theorem 5.8.2 say about the limiting distribution of the Bayes rule for squared error loss (assuming that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \equiv P$ with $\theta_0 \in (0, \infty)$)?

3. Consider Example 5.5.4 on pages 16 and 17 of the Chapter 5 notes.
(a) Show that the variance of $\hat{\psi}$ is given by

$$Var(\hat{\psi}_n) = \frac{1}{n}\left\{\frac{1}{B}\sum_{j=1}^{B}\frac{\theta_j}{\xi_j} - \psi(\theta)^2\right\}.$$

[Hint: use the formula $Var(Y) = EVar(Y|X) + Var[E(Y|X)]$ twice.]
(b) Use the result of (a) to show that

$$Var(\hat{\psi}_n) \leq \frac{1}{n\delta}$$

under the assumption that $\xi_j \geq \delta > 0$ for all $1 \leq j \leq B$.

4. Lehmann and Casella, TPE, Problem 5.17, page 293. (Also note Problems 5.18, 5.19, 5.20, page 293.)

5. **Optional bonus problem 1:** Lehmann and Casella, TPE, Problem 5.20, pages 293-294.

6. **Optional bonus problem 2:** (a) Suppose that $X \sim F$, and let $m = F^{-1}(1/2)$, $\mu = E(X)$, $\sigma^2 = Var(X)$, and we assume that $E(X^2) < \infty$. Show that

$$|m - \mu| \leq \sqrt{2\sigma^2}\,.$$

Hint: use Chebychev's inequality.
(b) Let m and μ be the median and mean of F as in A, and let M be the *mode* of F, assuming that it is well-defined: i.e. we suppose that F has density f which is strictly increasing to the left of M and strictly decreasing to the right of M. Show that $\mu \leq m \leq M$ if there is an x_0 such that

$$f(m+x) - f(m-x)\begin{cases} \geq 0 & \text{for } \ 0 \leq x < x_0 \\ \leq 0 & \text{for } \ x_0 < x < \infty \end{cases}. \tag{1}$$

If the inequalities in (1) are reversed, then $M \leq m \leq \mu$. Hint: show that

$$m - \mu = \int_0^\infty \{F(m-x) + F(m+x) - 1\}dx\,.$$

(c) Examine (1) and the conclusion for the distributions Gamma$(r, 1)$ with $r = 1, 2$.