

Statistics 582, Problem Set 5

Wellner; 2/7/2007

Reading: Chapter 5, sections 6-8

Due: Wednesday, February 14, 2007.

Reminder: Midterm Exam: Friday, February 9.

1. Suppose that $X_n \equiv X \sim \text{Multinomial}_k(n, \underline{\theta})$.
 - (a) Suppose that the prior distribution on θ is given by a Dirichlet distribution, $\text{Dirichlet}(\underline{\alpha})$:

$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1-1} \cdots \theta_k^{\alpha_k-1} 1_{[\underline{\theta}: \sum \theta_i=1]}.$$

Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4

- (b) What is the posterior distribution for θ ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
 - (c) Find a minimax estimator d_M of $\underline{\theta}$.
2. Find the limit distribution of the minimax estimator d_M in problem 1 (i.e. $\sqrt{n}(d_M(X_n) - p) \rightarrow_d$ “something” and find “something”). Is d_M a regular estimator of p ?
 3. Let $\Theta = (0, \infty)$, $\mathbf{A} = [0, \infty)$, let X have the discrete distribution

$$p(x, \theta) = \binom{r+x-1}{x} \theta^x (\theta+1)^{-(r+x)}, \quad x = 0, 1, 2, \dots$$

where r is some known positive integer; this is the negative binomial distribution reparametrized so that $E_\theta X = r\theta$. Suppose that

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta(\theta + 1)}.$$

- (a) Show that the usual estimator, $d_0(X) = X/r$ is an equalizer rule; i.e. show that it has a risk function $R(\theta, d_0)$ which is constant in θ .
- (b) Show that the usual estimator d_0 is generalized Bayes with respect to Lebesgue measure on $(0, \infty)$ provided $r > 1$. (A generalized Bayes rule is a rule that minimizes the posterior Bayes risk even when starting with an improper prior; see e.g. Ferguson, MS, page 50.) (What happens if $r = 1$?)

(c) Find Bayes decision rules with respect to the prior distributions $\Lambda_{\alpha,\beta}$ with densities

$$\lambda_{\alpha,\beta}(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (\theta + 1)^{-(\alpha+\beta)} 1_{(0,\infty)}(\theta),$$

the distribution of $\theta = Z/(1 - Z)$ where $Z \sim \text{Beta}(\alpha, \beta)$.

(d) Show that $d(X) = X/(r + 1)$ is minimax. [Note that d_0 is not minimax, hence not admissible.]

4. (a) Let $(X|\sigma^2) \sim N(0, \sigma^2)$. Show that the conjugate prior for σ^2 is the distribution of $1/Y$ where Y has a gamma distribution.
- (b) Suppose that $(X|\theta, \kappa) \sim N(\theta, 1/\kappa)$, $(\theta|\kappa) \sim N(\mu, \tau/\kappa)$, and $\kappa \sim \text{Gamma}(\alpha, \beta)$. Show that the posterior distribution of (θ, κ) has the same form as the prior.
- (c) Find the marginal posterior distribution for θ in B.
- (d) If X_1, \dots, X_n are i.i.d. as X in (a), find the limiting distribution of the Bayes estimator of θ .

5. **Optional bonus problem:** Problem 3.9, Lehmann and Casella, TPE, page 286.