

Statistics 582, Problem Set 8 Solutions

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1. Continuation of problem 1, problem set 7: For P_0 and P_1 as given in problem 2 of problem set # 7, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1) = \int \sqrt{p_0 p_1} d\mu$. For the product laws P_{0n} and P_{1n} (corresponding to observation of X_1, \dots, X_n i.i.d. P_0 or P_1 respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20, 50, 100$. What does this imply about the test, ϕ_n say, based on X_1, \dots, X_n which minimizes the sum of risks?

Solution: First,

$$d_{TV}(P_0, P_1) = (1/2)\{.2 + .1 + .1 + .2\} = (1/2)(.6) = .3.$$

Furthermore,

$$\rho(P_0, P_1) = \sqrt{.08} + \sqrt{.02} + \sqrt{.06} + \sqrt{.08} = .9520558\dots$$

so that $H^2(P_0, P_1) = 1 - \rho(P_0, P_1) = .0479442\dots$, and $H(P_0, P_1) = .2189616\dots$. Note that the inequalities of problem xxx are indeed satisfied:

$$.0479442\dots < .3 < .2189616(2 - .0479442\dots)^{1/2} = .30592244\dots < .3096585\dots$$

For $n = 20, 50, 100$ we have:

n	$\rho(P_0^n, P_1^n)$	$H(P_0^n, P_1^n)$	$H^2(P_0^n, P_1^n)$
1	.952056	.2189616	.0479
20	.374326	.7909956	.626
50	.085728	.9561755	.915
100	.007349	.9963185	.993

Since the test $\phi = \phi(\underline{X})$ which minimizes the sum of risks has

$$\begin{aligned} E_0\phi(\underline{X}) + E_1(1 - \phi(\underline{X})) &= \int p_0(\underline{x}) \wedge p_1(\underline{x}) d\mu(\underline{x}) \\ &\leq \rho(P_0^n, P_1^n) \\ &= \rho^n(P_0, P_1) \rightarrow 0. \end{aligned}$$

From the table above we see that this happens quite rapidly.

2. Suppose that X_1, \dots, X_n are i.i.d. $N(\theta, \sigma^2)$.

A. Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ . Find the entire risk body and plot it.

B. What happens to the risk body as n grows or as $\sigma_0 \rightarrow 0$?

C. What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?

Solution: A. If X_1, \dots, X_n are i.i.d. $N(\theta, \sigma_0)$, to find optimal tests ϕ we can reduce (by sufficiency) to consideration of $\bar{X} \sim N(\theta, \sigma_0^2/n)$. My favorite family of tests (in fact the most powerful tests) of H versus K are the tests $\phi_c(\underline{X}) = 1\{\bar{X} > c\}$. For these tests

$$\begin{aligned} R(0, \phi_c) &= E_0 \phi_c(\underline{X}) = P_0(\bar{X} > c) \\ &= P_0(\sqrt{n}(\bar{X} - 0)/\sigma_0 > \sqrt{nc}/\sigma_0) \\ &= 1 - \Phi(\sqrt{nc}/\sigma_0) \end{aligned}$$

and

$$\begin{aligned} R(1, \phi_c) &= E_1(1 - \phi_c(\underline{X})) \\ &= P_1(\bar{X} \leq c) = P_1(\sqrt{n}(\bar{X} - 1) \leq \sqrt{n}(c - 1)) \\ &= \Phi(\sqrt{n}(c - 1)/\sigma_0). \end{aligned}$$

Since these tests are MP for testing H versus K , there are no points with risks below the curve given by $\{(R(0, \phi_c), R(1, \phi_c)) : c \in \mathbb{R}\}$; this is the lower boundary of the risk body. Note that the tests $\phi_{\text{ignore}}(\underline{X}) \equiv \alpha$ have risks $R(0, \phi_{\text{ignore}}) = \alpha$, $R(1, \phi_{\text{ignore}}) = 1 - \alpha$. Thus the line $\{(\alpha, 1 - \alpha) : \alpha \in [0, 1]\}$ is in the risk body. Furthermore, note that the tests $\phi'_c(\underline{X}) \equiv 1 - \phi_c(\underline{X}) = 1\{\bar{X} \leq c\}$ are MP for testing $H : \theta = 0$ versus $K' : \theta = \theta_1 < 0$, and by the Karlin - Rubin theorem these tests minimize the power function at points $\theta = \theta_1$ in the class of all tests with fixed power function (say at α) at $\theta = \theta_0$. Since

$$\text{Power}_{\phi'_c}(\theta) = E_{\theta} \phi'_c = 1 - R(\theta, \phi_c),$$

this says that the tests ϕ'_c maximize $R(1, \phi_c)$ over tests ϕ with $R(0, \phi) = \alpha$. Hence there are no points in the risk body with risks above the curve given by $\{(1 - R(0, \phi_c), 1 - R(1, \phi_c)) : c \in \mathbb{R}\}$.

B. As n grows or $\sigma_0 \rightarrow 0$ the risk body expands out toward the boundary of the square $[0, 1]^2$; see the plots below.

C. As $\theta_1 \rightarrow \theta_0 = 0$, the risk body contracts toward the diagonal line $(\alpha, 1 - \alpha)$ – since the testing problem becomes harder. See the plots below.

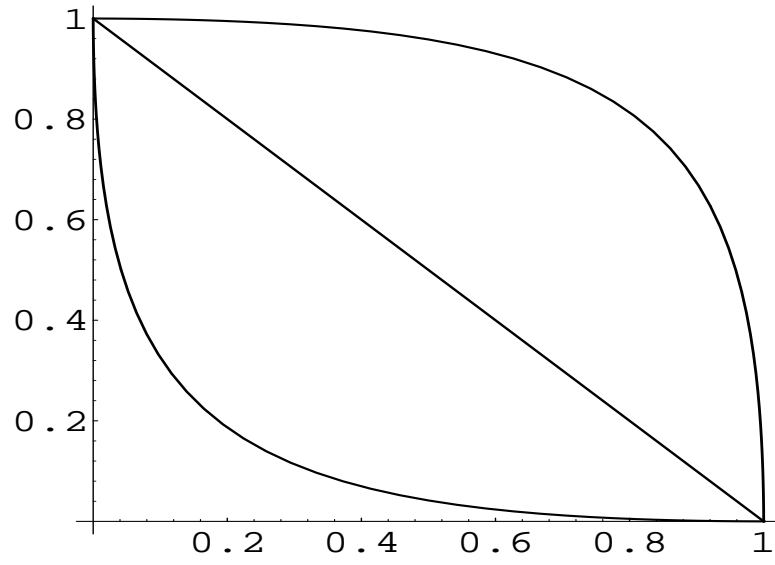


Figure 1: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = 1$

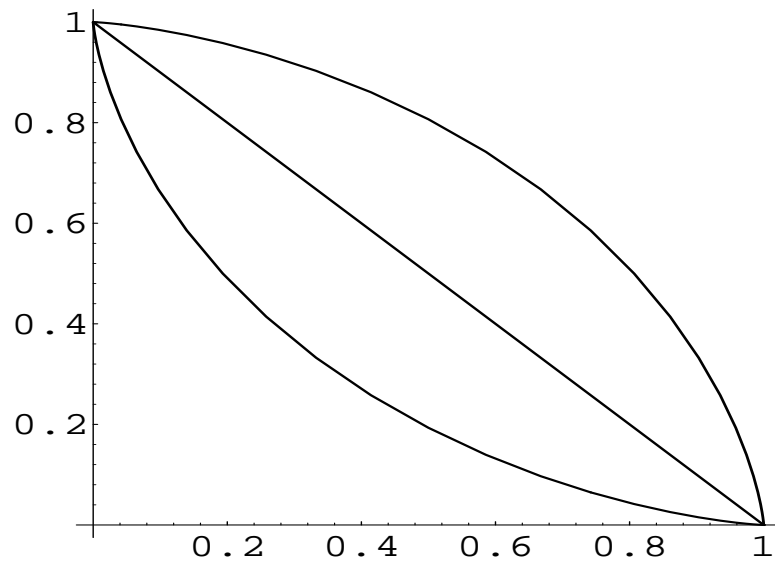


Figure 2: Risks for normal mean test, $n = 3$, $\sigma_0 = 1$, $\theta_1 = .5$

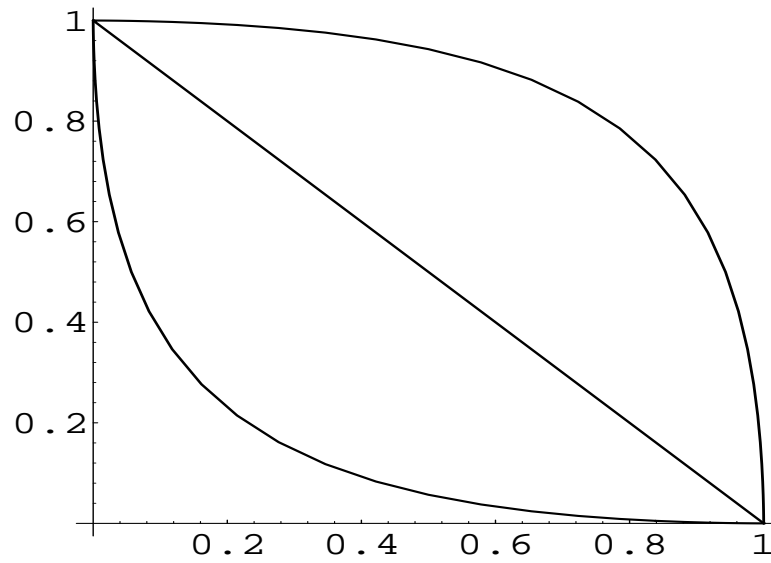


Figure 3: Risks for normal mean test, $n = 10$, $\sigma_0 = 1$, $\theta_1 = .5$

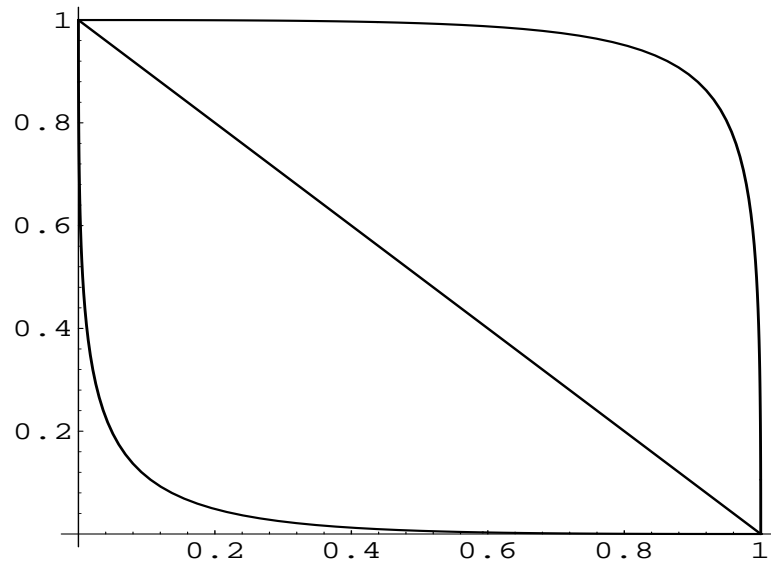


Figure 4: Risks for normal mean test, $n = 25$, $\sigma_0 = 1$, $\theta_1 = .5$

3. Let X_1, \dots, X_n be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
 - (a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

(b) Find the subclass of the tests that are independent of θ_1 . These are UMP tests of H_0 versus $H'_1 : \theta > \theta_0$.

(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is UMP of size α for testing $H'_0 : \theta \leq \theta_0$ versus $H'_1 : \theta > \theta_0$ but that ϕ is not admissible.

(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \leq b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \neq \theta_0$.

Solution: (a) For testing $\theta = \theta_0$ versus $\theta = \theta_1 > \theta_0$, the class of all NP tests is given by tests of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) > k \theta_0^{-n} 1_{[0, \theta_0]}(t) \\ \gamma(t), & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) = k \theta_0^{-n} 1_{[0, \theta_0]}(t) \\ 0, & \text{if } \theta_1^{-n} 1_{[0, \theta_1]}(t) < k \theta_0^{-n} 1_{[0, \theta_0]}(t) \end{cases} .$$

Thus for $k = (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \leq \theta_1 \\ \gamma(t), & \text{if else;} \end{cases}$$

for $k > (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t \leq \theta_1 \\ \gamma(t), & \text{if else} \\ 0, & \text{if } 0 \leq t \leq \theta_0; \end{cases}$$

and for $k < (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \theta_1 \\ \gamma(t), & \text{if else.} \end{cases}$$

(b) The subclass of these tests that do not depend on θ_1 is the class of tests ϕ with

$$\phi(t) = \begin{cases} 1, & \text{if } \theta_0 < t < \infty \\ \gamma(t), & \text{if else} \end{cases}$$

where $E_{\theta_0} \gamma(T) = \alpha$.

(c) The test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is of the form of the tests in (b) with $\gamma(t) = \alpha$ for $0 \leq t \leq \theta_0$, and hence is UMP of size α for testing $\theta = \theta_0$ versus $\theta > \theta_0$. To see that it is UMP for testing $\theta \leq \theta_0$ versus $\theta > \theta_0$, we first compute its power function to confirm that it is of size α for the composite null hypothesis $\theta \leq \theta_0$. The power is

$$\begin{aligned} \beta_\phi(\theta) &= E_\theta \phi(T) = P_\theta(T > \theta_0) + \alpha P_\theta(T \leq \theta_0) \\ &= \{1 - (\frac{\theta_0}{\theta})^n\} 1_{(\theta_0, \infty)}(\theta) + \alpha 1_{[0, \theta_0]}(\theta) + \alpha \left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0, \infty)}(\theta) \\ &= \{1 - (\frac{\theta_0}{\theta})^n (1 - \alpha)\} 1_{(\theta_0, \infty)}(\theta) + \alpha 1_{[0, \theta_0]}(\theta). \end{aligned}$$

Thus we see that $\beta_\phi(\theta) = \alpha$ for $\theta \leq \theta_0$, and ϕ is of size α for $\theta \leq \theta_0$. Since the class of size α tests for testing $\theta = \theta_0$ is a larger class than the class of size α tests for testing $\theta \leq \theta_0$, and since ϕ is UMP in the larger class, it is also UMP in the smaller class. Hence ϕ is UMP for testing $\theta \leq \theta_0$ versus $\theta > \theta_0$. But the competing test $\phi_0(t) = 1\{t > (1 - \alpha)^{1/n}\theta_0\}$ has power function

$$\begin{aligned}\beta_{\phi_0}(\theta) &= E_\theta \phi_0(T) = P_\theta(T > \theta_0(1 - \alpha)^{1/n}) \\ &= 1 - \left\{ (1 - \alpha) \left(\frac{\theta_0}{\theta} \right)^n 1_{(\theta_0(1 - \alpha)^{1/n}, \infty)}(\theta) + 1_{[0, \theta_0(1 - \alpha)^{1/n}]}(\theta) \right\},\end{aligned}$$

so the power function of the test ϕ_0 is strictly below that of the test ϕ on the set $[0, \theta_0)$. Hence ϕ is inadmissible and the test ϕ_0 is also UMP.

(d) The test $\phi(t) = 1 - 1_{(\theta_0\alpha^{1/n}, \theta_0]}(t)$ is of size α for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$ since

$$E_{\theta_0} \phi(T) = P_{\theta_0}(T \leq \theta_0\alpha^{1/n}) = \left(\frac{\theta_0\alpha^{1/n}}{\theta_0} \right)^n = \alpha.$$

Furthermore it is of the form of the class of all UMP tests for testing $\theta = \theta_0$ versus $\theta > \theta_0$, hence it UMP among the size α tests for $\theta > \theta_0$. For testing $\theta = \theta_0$ versus $\theta = \theta_1 < \theta_0$, the NP Pearson tests of the form $\phi_1(t) = \gamma(t)1_{[0, \theta_0]}(t)$ are most powerful of their size. But the test ϕ is of this form (with $\gamma(t) = 1_{[0, \theta_0\alpha^{1/n}]}(t)$), is in this class and does not depend on $\theta_1 < \theta_0$. Hence ϕ is UMP for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$. [This is an unusual situation in which we get “something for free” from the structure of the uniform distributions. Usually two-sided tests are *not* UMP!]

4. Show that the Cauchy scale family of distributions given by

$$p_\theta(x) = \frac{1}{\pi\theta} \frac{1}{1 + (x/\theta)^2}$$

does not have monotone likelihood ratio, but that the distribution of the sufficient statistic $|X|$ (where $X \sim P_\theta$ with Cauchy density p_θ as in the last display) does have monotone likelihood ratio.

Solution: For $\theta' > \theta$ the ratio $p_{\theta'}/p_\theta$ is given by

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = \frac{\theta' \theta^2 + x^2}{\theta \theta'^2 + x^2}$$

which increases from $\theta/\theta' < 1$ to $\theta'/\theta > 1$ as x^2 increases from 0 to ∞ . Thus $\{p_\theta\}$ has MLR in $T(x) = x^2$ (or $|x|$).

Alternatively, the sufficient statistic $|X|$ has density

$$p_{\theta}^{|X|}(y) = 2p_{\theta}(y)1_{[0,\infty)}(y) = \frac{2}{\pi\theta} \frac{1}{1 + (y/\theta)^2} 1\{y \geq 0\},$$

which has MLR in y .