Statistics 582, Problem Set 7 Solutions

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1. Continuation of problem 2, problem set 5: Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have distribution P_θ and density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on $\mathbb{R}$, and that $\theta \sim \Gamma(\alpha, \beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

In problem set 5 we found the Bayes rules with respect to squared error loss $L(\theta, a) = (\theta - a)^2$ and weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$.

A. Prove a (conditional) limit theorem for the posterior distributions given $\underline{X}$.
B. What does theorem 5.8.2 say about the limiting distribution of the Bayes rule for squared error loss (assuming that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \equiv P$ with $\theta_0 \in (0, \infty)$)?

Solution: A. Now

$$\begin{aligned} \theta \sim \text{Gamma}(\alpha + n, \beta + \sum X_i) &=_d (\beta + \sum X_i)^{-1} \text{Gamma}(\alpha + n, 1) \\ &=_d (\beta + \sum X_i)^{-1} (Y_0 + \sum_{i=1}^n Y_i) \end{aligned}$$

where $Y_0 \sim \text{Gamma}(\alpha, 1)$, and $Y_i \sim \text{Gamma}(1, 1) = \text{Exp}(1)$, $i = 1, \ldots, n$ are all independent. Thus conditionally on the X_i's we have, with $Z \sim N(0, 1)$ and with θ_0 the true value of θ,

$$\begin{aligned} \sqrt{n}(\theta - E(\theta|\underline{X})) &=_d \sqrt{n} \frac{Y_0 + \sum_{i=1}^n Y_i - (\alpha + n)}{\beta + \sum_{i=1}^n X_i} \\ &= \sqrt{n}(\overline{Y}_n - 1) \frac{1}{\overline{X}_n + n^{-1}\beta} + \sqrt{n}(Y_0 - \alpha) \frac{1/n}{\overline{X}_n + n^{-1}\beta} \\ &\to_d Z \frac{1}{\theta_0^{-1}} \sim N(0, \theta_0^2) \end{aligned}$$

almost surely with respect to the distribution of $X_1, X_2, \ldots$. Note that the posterior mean $E(\theta|\underline{X})$ can be replaced here by either the MLE $1/\overline{X}_n$ or by $T_n = \theta_0 + (nI(\theta_0))^{-1} \sum_{i=1}^n \dot{l}_\theta(X_i) = 2\theta_0 - \theta_0^2 \overline{X}_n$ since

$$\sqrt{n}(E(\theta|\underline{X}) - 1/\overline{X}_n) = o_p(1)$$

and similarly with T_n in place of $1/\overline{X}_n$.
B. In the present case Theorem 5.8.2 says that

$$\sqrt{n}(E(\theta|\underline{X}) - \theta_0) \to_d N(0, 1/I(\theta_0)) = N(0, \theta_0^2)$$

since $I(\theta_0) = 1/\theta_0^2$.

2. A random variable X takes on the values $1,\ 2,\ 3,\ 4$ with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.2	.1	.3	.4
$p_1(x)$	.4	.2	.2	.2

A. Find a most powerful test of size $\alpha = .2$ for testing p_0 versus p_1 and determine its power.
B. Find a test ϕ which minimizes the sum of risks $a + b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$.

Solution: Now $p_1(x)/p_0(x) = 2,\ 2,\ 2/3,\ 1/2$, according as $x = 1,\ 2,\ 3,\ 4$, so a MP test of size $\alpha = .2$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x = 2, 3, 4. \end{cases}$$

Then

$$E_0\phi(X) = P_0(X = 1) = .2$$

while

$$\text{Power} = E_1\phi(X) = P_1(X = 1) = .4.$$

Note that any test ϕ^* of the form

$$\phi^*(x) = \begin{cases} \gamma(x), & \text{if } x = 1, 2 \\ 0, & \text{if } x = 3, 4\ . \end{cases}$$

is also most powerful of size α in this case as long as we choose γ so that $E_0\gamma(X)1_{\{1,2\}}(X) = .2$. The test ϕ above is of the form ϕ^* with $\gamma(x) = 1\{X = 1\}$. Another choice would be $\gamma(x) = 2/3$: then $E_0\phi^*(X) = (2/3)(P_0(X = 1) + P_0(X = 2)) = (2/3)(.2 + .1) = .2$, and the power is

$$E_1\phi^*(X) = (2/3)(.4 + .2) = .4.$$

B. A test which minimizes $a + b$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } p_1(x) \ge p_0(x) \\ 0, & \text{if } p_1(x) < p_0(x) \end{cases}$$
$$= \begin{cases} 1, & \text{if } x = 1,\ 2 \\ 0, & \text{if } x = 3, 4 \end{cases}$$

Then $a = E_0\phi(X) = .3$ and $b = E_1(1 - \phi(X)) = .4$, and hence $(a + b)_{\min} = .7$. Note that $\int p_0 \wedge p_1 d\mu = .7$, so this result agrees with the solution of problem 4 of problem set # 4.

3. (Problem 3.6, Lehmann and Romano, TSH, page 93.) Suppose that P_0, P_1, and P_2 be the probability distributions assigning to the integers $1, \ldots, 6$ the following probabilities:

x	1	2	3	4	5	6
$p_0(x)$	.03	.02	.02	.01	0	.92
$p_1(x)$	.06	.05	.08	.02	.01	.78
$p_2(x)$	.09	.05	.12	0	.02	.72

Determine whether there exists a level-α test of $H: \; P = P_0$ which is UMP against the alternatives P_1 and P_2 when:
(i) $\alpha = .01$; (ii) $\alpha = .05$; (iii) $\alpha = .07$.

Solution: Here the table of likelihood ratios is as follows:

x	1	2	3	4	5	6
$p_1(x)/p_0(x)$	2	5/2	4	2	∞	78/98
$p_2(x)/p_0(x)$	3	5/2	6	0	∞	72/98

(i) For $\alpha = .01$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1\{x = 5\} + (1/2)1\{x = 3\},$$
$$\phi_2(x) = 1\{x = 5\} + 1/2)1\{x = 3\},$$

so $\phi_1 = \phi_2$ is Uniformly most powerful.
(ii) For $\alpha = .05$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,3,5\}},$$

so there is no UMP test of P_0 versus P_1 and P_2 at this level.
(iii) For $\alpha = .07$, the most powerful tests of P_0 versus P_1 and P_2 are of the form

$$\phi_1(x) = 1_{\{2,3,5\}}(x) + \gamma(x)1_{\{1,4\}}(x),$$
$$\phi_2(x) = 1_{\{1,2,3,5\}},$$

so by taking $\gamma(x) = 1\{x = 1\}$, $\phi_1(x) = \phi_2(x)$, and this test is Uniformly Most Powerful for testing P_0 versus P_1 and P_2.

4. (Problem 3.7, Lehmann and Romano, TSH, page 94.) Suppose that the distribution of X is given by

x	0	1	2	3
$p_\theta(x)$	θ	2θ	$.9 - 2\theta$	$.1 - \theta$

where $0 < \theta < .1$. For testing $H : \theta = .05$ against $\theta > .05$ at level $\alpha = .05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = .5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

Solution: The likelihood ratios $P_{\theta'}(X = x)/P_\theta(X = x)$

x	0	1	2	3
$P_\theta(X = x)$	θ	2θ	$.9 - .2\theta$	$.1 - \theta$
$\frac{P_{\theta'}(X=x)}{P_\theta(X=x)}$	$\frac{\theta'}{\theta}$	$\frac{\theta'}{\theta}$	$\frac{9-20\theta'}{9-20\theta}$	$\frac{1-10\theta}{1-10\theta'}$

It is easy to check that

$$\frac{\theta'}{\theta} > \frac{9 - 20\theta'}{9 - 20\theta} > \frac{1 - 10\theta'}{1 - 10\theta}$$

Hence this family has monotone decreasing likelihood ratio in x (though not strictly), and strictly decreasing likelihood ratio in

$$\begin{aligned} T(x) &= 1\{x = 0\} + 1\{x = 1\} + 2 \cdot 1\{x = 2\} + 3 \cdot 1\{x = 3\} \\ &= x1\{x > 0\} + 1\{x = 0\}. \end{aligned}$$

It follows from the Karlin - Rubin theorem that a UMP test of $H : \theta \le \theta_0 = .05$ (of its level) is given by

$$\phi(X) = 1_{[T(X)<k]} + \gamma(X)1_{[T(X)=k]}. \quad (1)$$

(i) Note that the test $\phi_1(X) = 1\{X = 0\}$ is of the form (1) with $k = 1$ and $\gamma(X) = 1\{X = 0\}$ and it has level $\alpha = .05$; hence it is a UMP test of H versus K. The power of ϕ_1 is given by $\beta_1(\theta) \equiv E_\theta \phi_1(X) = \theta$.
(ii) The test $\phi_2(X) = .51\{X = 1\}$ is also of the form (1) with $k = 1$ and $\gamma(X) = .5 \cdot 1\{X = 1\}$ and it has level $\alpha = .05$. Hence it is also a UMP test of H versus K. The power of ϕ_2 is given by $\beta_2(\theta) \equiv E_\theta \phi_2(X) = \theta$.
(iii) The test $\phi_3(X) = 1\{X = 3\}$ is clearly not of the form (1). It has power function $\beta_3(\theta) = E_\theta \phi_3(X) = .1 - \theta$, so $\beta_3(.05) = .05$, but $\beta_3(\theta) > .05$ for $\theta < .05$ while $\beta_3(\theta) < .05$ for $\theta > \theta_0 = .05$. In fact, this is a UMP test of $\tilde{H} : \theta \geq \theta_0$ versus $\tilde{K} : \theta < \theta_0$.