Statistics 582, Final Exam Solutions

Wellner; 3/14/2006

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A uniformly most powerful level α test.

   (b) An unbiased test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (c) A similar test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A level α permutation test.

   (e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.

   Solution: see the class notes.

2. (30 points) **State** *any three* of the following results:

   (a) A theorem relating similar tests to tests with Neyman structure.

   (b) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   (c) A theorem about admissibility properties of the sample mean $\overline{X}$ when sampling from a normal distribution on $\mathbb{R}$ and a contrasting theorem for sampling from a normal distribution on $\mathbb{R}^d$.

   (d) Stein's identity for $Eg'(X)$ where $X \sim N_1(0, \sigma^2)$ and $E|g'(X)| < \infty$.

   (e) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.

   (e) A conditional limit theorem about the large sample behavior of posterior distributions

   Solution: see the class notes.

3. (40 points) **State** and **prove** the Neyman - Pearson lemma.

   Solution: see the class notes.

4. (48 points) Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. Consider testing $H: \ p_2 \le p_1$ versus $K: \ p_2 > p_1$.

   (a) Write the joint density (probability mass function) $p(x, y; p_1, p_2) \equiv P_{p_1,p_2}(X = x, Y = y)$ in exponential family form $c(\theta, \xi)\exp(\theta U(x, y) + \xi T(x, y))h(x, y)$ where $T(x, y)$ is sufficient for the boundary $\Theta_B = \{(p_1, p_2) \in [0, 1]^2 : \ p_1 = p_2\}$ and $U(x, y) = y$.

(b) Show carefully that testing H versus K is equivalent to testing $H_1 : \theta \le 0$ ($\xi =$ anything) versus $K_1 : \theta > 0$ ($\xi =$ anything).

(c) What is the conditional distribution of $U(X, Y) = Y$ given $T = t$ under $p_1 = p_2$? Compute it explicitly and give its name.

(d) Find the UMP unbiased test of size α as explicitly as possible when $m = 3$, $n = 2$, $t = 2$, and $\alpha = 1/10$.

(e) Relate the conditional distribution of the test statistic involved in (c) to a problem involving sampling without replacement from a finite population. Identify the contents of the urn (i.e. the numbers on the balls in the urn) and calculate the mean and variance of Y given $T = t$ in this conditional distribution.

(f) Use the results of (e) together with the Wald-Wolfowitz-Noether-Hájek CLT to show a conditional (on $T = t$) CLT for Y appropriately centered and normalized if $0 < \liminf(m/N) \le \limsup(m/N) < 1$. [Make sure that you verify the key hypothesis of the theorem.]

Solution: (a) Now

$$
\begin{aligned}
&p(x, y; p_1, p_2)\\
&= \binom{m}{x} p_1^x (1-p_1)^{m-x} \binom{n}{y} p_2^y (1-p_2)^{n-y}\\
&= (1-p_1)^m (1-p_2)^n \exp\left(y \log\left(\frac{p_2/(1-p_2)}{p_1/(1-p_1)} \right) + (x+y)\log(p_1/(1-p_1)) \right) \binom{m}{x}\binom{n}{y}\\
&= c(\theta, \xi) \exp(\theta U(x, y) + \xi T(x, y)) h(x, y)
\end{aligned}
$$

where

$$
\begin{aligned}
U(x, y) &= y, & \theta &= \theta(p_1, p_2) = \log\left(\tfrac{p_2/(1-p_2)}{p_1/(1-p_1)} \right)\\
T(x, y) &= x + y, & \xi &= \xi(p_1, p_2) = \log(p_1/(1-p_1)),
\end{aligned}
$$

(b) Since $\theta = 0$ on the boundary set $p_1 = p_2$, and since $\theta(p_1, p_2)$ is a monotone increasing function of p_2 for each fixed p_1, it follows that $H: \ p_2 \le p_1$ versus $K: \ p_2 > p_1$ corresponds to $H_1 : \theta \le 0$ versus $K_1 : \theta > 0$.
(c) The conditional distribution of $U = Y$ given $T = X + Y$ under $p_1 = p_2$ is

given by

$$
\begin{aligned}
p(y|t) &= P_{(p_1,p_1)}(Y=y|T=t) = \frac{P_{(p_1,p_1)}(Y=y, T=X+Y=t)}{P_{(p_1,p_1)}(T=t)} \\
&= \frac{P_{(p_1,p_1)}(Y=y, X=t-y)}{P_{(p_1,p_1)}(T=t)} = \frac{\binom{m}{t-y} p_1^{t-y}(1-p_1)^{m-(t-y)} \binom{n}{y} p_1^y (1-p_1)^{n-y}}{\binom{N}{t} p_1^t (1-p_1)^{N-t}} \\
&= \frac{\binom{n}{y}\binom{m}{t-y}}{\binom{N}{t}}, \qquad \text{for } \ 0 \vee (t-m) \le y \le t \wedge n
\end{aligned}
$$

where $N \equiv m+n$. This is the hypergeometric distribution with parameters N, n, t (total number of balls N, number of "white balls" n (and number of black balls $m = N - n$), and number of draws t), describing the distribution of the total number of white balls drawn in a sample of size t drawn without replacement from an urn containing n white balls and $m = N - n$ black balls.

(d) The UMP unbiased test of H versus K rejects when $Y > c_t$ where c_t is determined from the hypergeometric distribution in (c) so that $P(Y > c_t | T = t) = \alpha$ (or a randomized version of this) When $m = 3$, $n = 2$, $t = 2$, and $\alpha = .1$, the hypergeometric distribution become

$$
\begin{aligned}
P(Y=2|T=2) &= \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}, \\
P(Y=1|T=2) &= \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10}, \\
P(Y=0|T=2) &= \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10},
\end{aligned}
$$

so $c_t = 1$ and rejecting when $Y > 1$ (i.e. when $Y = 2$) gives conditional level $\alpha = .1$.

(e) For the description, see the solution of (c) above; we can also identify the "white balls" as those with the number 1, and the "black balls" as those with the number zero. Then $E(Y|T=t) = t(n/N)$ while $Var(Y|T=t) = (1 - \frac{t-1}{N-1}) t\sigma_z^2$ where

$$
\begin{aligned}
\sigma_z^2 &= \frac{1}{N}\sum_1^N (z_i - \overline{z})^2 = \frac{1}{N}\{n(1-n/N)^2 + m(0-n/N)^2\} \\
&= (n/N)(1-n/N)^2 + (m/N)(0-n/N)^2 \\
&= (n/N)(1-n/N)\{(1-n/N) + n/N\} = (n/N)(1-(n/N)).
\end{aligned}
$$

(f) Since we can view the conditional distribution of Y given $T = t$ as the total in drawing a sample of size t balls without replacement from an urn with balls labeled with z_i's, n of which are 1's and m of which are 0's, and by the calculation of the conditional mean and variance in (e),

$$\sigma_N^2 = Var[(Y/T)|T=t] = \left(1 - \frac{t-1}{N-1}\right)\frac{\sigma_z^2}{t} = \left(1 - \frac{t-1}{N-1}\right)\frac{(n/N)(1-n/N)}{t}$$

it follows from the W-W-N-H finite sampling CLT that with $\bar{Y} = Y/t$

$$\frac{\bar{Y} - \mu_z}{\sigma_N} = \frac{Y/t - (n/N)}{\sigma_N} \to_d N(0,1)$$

as $t \to \infty$ and $N \to \infty$ if $0 < \liminf((N-t)/N) \le \limsup((N-t)/N) < 1$ and the Noether condition holds. But the latter is easily satisfied in this case since

$$\begin{aligned} \eta_N = \frac{\max_{i \le N} |z_i - \overline{z}|^2}{\sum_1^N (z_i - \overline{z})^2} &= \frac{|1-(n/N)|^2 \vee |0-(n/N)|^2}{n(1-(n/N))^2 + m(0-n/N)^2} \\ &= \frac{(m/N) \vee (n/N)}{m \wedge n} \le \frac{1}{m \wedge n} \to 0 \end{aligned}$$

as $m \wedge n \to \infty$. Note that

$$0 < \liminf((N-T)/N) \le \limsup((N-T)/N) < 1 \tag{1}$$

holds almost surely if and only if

$$0 < \liminf(T/N) \le \limsup(T/N) < 1, \tag{2}$$

and since $T/N = (X+Y)/N = (m/N)(X/m) + (n/N)(Y/n)$ where $X/m \to_{a.s.} p_1$ and $Y/n \to_{a.s.} p_2$, (1) and (2) hold if $0 < p_1 \vee p_2$ and $p_1 \wedge p_2 < 1$ and $0 < \liminf(n/N) \le \limsup(n/N) < 1$.

5. (36 points) A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.1	.05	.35	.5
$p_1(x)$	.3	.25	.35	.1

(a) For the usual $0-1$ loss, find a most powerful test of size .10 for testing $H: p = p_0$ versus $K: p = p_1$ and determine its power.

(b) Find a test ϕ which minimizes the sum of risks $E_0\phi + E_1(1-\phi)$.

(c) If the losses are $L(1,1) = L(0,0) = 0$, $L(0,1) = 10$, $L(1,0) = 5$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.7, .3)$, find the Bayes rule d_B and the minimax rule d_M.

Solution:

(a). The ratios of the probabilities under the two hypotheses are given by the following table

x	1	2	3	4
$p_0(x)$	.1	.05	.35	.5
$p_1(x)$	.3	.25	.35	.1
$p_1(x)/p_0(x)$	3	5	1	1/5

Thus by the Neyman - Pearson lemma the most powerful test of size .10 is $\phi(x) = 1\{x = 2\} + (.5)1\{x = 1\}$. The power of this test is $\beta_\phi = E_1\phi(X) = P_1(X = 2) + .5P_1(X = 1) = .25 + (.5)(.3) = .4$.

(b). The sum of risks $E_0\phi + E_1(1-\phi) = 1 + \sum_x \phi(x)\{p_0(x) - p_1(x)\}$ is minimized by any rule $\phi(x) = 1\{x : \ p_1(x) > p_0(x)\} = 1_{\{1,2\}}(x)$. The minimum sum of risks is $E_0\phi(X) = 1 + (p_0(1) - p_1(1)) + (p_0(2) - p_1(2)) = 1 - .2 - .2 = .6$. [Note that the total risk of the Neyman-Pearson test of size .10 is $.1 + .6 = .7$.]

(c). For a rule $d = (d_1, d_2, d_3, d_4)$ which chooses 1 with probability d_x when x is observed, the (ordinary) risks are

$$\begin{aligned}
R(0,d) &= E_0L(0,d) = 10\{d_1(.1) + d_2(.05) + d_3(.35) + d_4(.5) \\
&= \frac{1}{10}\{10d_1 + 5d_2 + 35d_3 + 50d_4\}, \\
R(1,d) &= E_1L(1,d) \\
&= 5\{(1-d_1)(.3) + (1-d_2)(.25) + (1-d_3)(.35) + (1-d_4)(.1)\} \\
&= 5 - 1.5d_1 - 1.25d_2 - 1.75d_3 - .5d_4.
\end{aligned}$$

Thus the Bayes risk for the prior $\lambda = (.7, .3)$ is

$$\begin{aligned}
\mathcal{R}(\lambda, d) &= .7R(0,d) + .3R(1,d) \\
&= \frac{.7}{10}\{10d_1 + 5d_2 + 35d_3 + 50d_4\} \\
&\qquad + 1.5 - (.3)(1.5d_1 + 1.25d_2 + 1.75d_3 + .5d_4) \\
&= 1.5 + (.7 - .45)d_1 + (.35 - .375)d_2 + (2.45 - .525)d_3 + (3.5 - .15)d_4.
\end{aligned}$$

Since the coefficients of d_1, d_3, and d_4 are positive, while the coefficient of d_2 is negative, the Bayes rule is given by $d = (0, 1, 0, 0)$ with corresponding Bayes risk $\mathcal{R}(\lambda, d_B) = 1.5 - .025 = 1.475$.

To find the minimax rule d_M, we equate the two ordinary risks to find

$$10d_1 + 5d_2 + 35d_3 + 50d_4 = 50 - 15d_1 - 12.5d_2 - 17.5d_3 - 5d_4$$

and solve for d_3 to find that

$$52.5d_3 = 50 - 25d_1 - 17.5d_2 - 55d_4,$$

or equivalently that

$$d_3 = \frac{50}{52.5} - \frac{25}{52.5}d_1 - \frac{17.5}{52.5}d_2 - \frac{55}{52.5}d_4.$$

Substitution into $R(0,d)$ yields a common risk of

$$\begin{aligned}
R(0,d) &= d_1 + .5d_2 + 5d_4 + \frac{35}{10}\left(\frac{50}{52.5} - \frac{25}{52.5}d_1 - \frac{17.5}{52.5}d_2 - \frac{55}{52.5}d_4\right) \\
&= \frac{35\cdot 5}{52.5} + \left(1 - \frac{35}{10}\frac{25}{52.5}\right)d_1 + \left(.5 - \frac{35}{10}\frac{17.5}{52.5}\right)d_2 + \left(5 - \frac{35}{10}\frac{55}{52.5}\right)d_4 \\
&= \frac{35\cdot 5}{52.5} + c_1d_1 + c_2d_2 + c_4d_4
\end{aligned}$$

where $c_1 < 0$, $c_2 < 0$ and $c_4 > 0$. Thus the minimax rule has $d_1 = d_2 = 1$, and $d_4 = 0$. This yields

$$d_3 = \frac{50}{52.5} - \frac{15}{52.5} - \frac{17.5}{52.5} = \frac{7.5}{52.5} = \frac{1}{7}.$$

Thus the minimax rule is $d_M = (1, 1, 1/7, 0)$, and the common risk is $R(0,d) = R(1,d) = 1 + .5 + (3.5)(1/7) = 2.0$.

Another way to find the minimax rule is to find a prior distribution $\lambda = (\lambda_0, 1-\lambda_0)$ which is least favorable: Note that with $\phi(x)$ now denoting the probability of choosing 1 when x is observed,

$$R(0,\phi) = 10E_0\phi(X), \qquad R(1,\phi) = 5E_0(1-\phi(X)),$$

and the Bayes risk is

$$\begin{aligned}
\mathcal{R}(\lambda,\phi) &= \lambda_0 10E_0\phi(X) + (1-\lambda_0)5E_1(1-\phi(X)) \\
&= 5(1-\lambda_0) + \int \phi\{\lambda_0 10p_0 - 5(1-\lambda_0)p_1\}d\mu,
\end{aligned}$$

so the Bayes rules with respect to the prior $(\lambda_0, 1-\lambda_0)$ are of the form

$$\begin{aligned}
\phi(x) &= 1\{10\lambda_0 p_0(x) < 5(1-\lambda_0)p_1(x)\} + \gamma 1\{10\lambda_0 p_1(x) = 5(1-\lambda_0)p_0(x)\} \\
&= 1\{p_1(x) > \frac{2\lambda_0}{1-\lambda_0}p_0(x)\} + \gamma 1\{p_1(x) = \frac{2\lambda_0}{1-\lambda_0}p_0(x)\}.
\end{aligned}$$

For the choice $\lambda_0 = 1/3$, this becomes

$$\phi(x) \quad = \quad 1\{p_1(x) > p_0(x)\} + \gamma 1\{p_1(x) = p_0(x)\},$$

with ordinary risks

$$R(0, \phi) = 10E_0\phi(X) = 10\{.15 + \gamma(.35)\},$$
$$R(1, \phi) = 5E_1(1 - \phi(X)) = 5\{.1 + (1 - \gamma).35\}.$$

Equating these two risks and solving for γ yields $\gamma = 1/7$. For this particular Bayes rule we have $R(0, \phi) = R(1, \phi) = 2 = \mathcal{R}(\lambda, \phi)$, and hence $\lambda = (1/3, 2/3)$ is a least favorable prior and $\phi(x) = 1_{\{1,2\}}(x) + (1/7)1_{\{3\}}(x)$ is minimax.

Do either problem 6 or problem 7.

6. (36 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. exponential(θ) random variables so that $1 - F_\theta(x) = \exp(-\theta x)$ for $x \geq 0$ and $\theta > 0$.
(a) Find the MLE of θ.
(b) If $\theta \sim \text{Gamma}(\alpha, \beta)$ so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta)$$

and $E(\theta) = \alpha/\beta$, find the posterior distribution of θ.
(c) Find the Bayes estimator of θ for squared error loss. Is it consistent?
(d) What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?

Solution: (a) The joint density of the data is

$$p(\underline{x}; \theta) = \prod_{i=1}^n \theta \exp(-\theta x_i) = \theta^n \exp(-\theta \sum_1^n x_i)$$

so $\log p(\underline{X}; \theta) = n \log \theta - \theta \sum_1^n X_i$, and we see that this is maximized by $\hat{\theta}_n = 1/\overline{X}$.
(b) The joint density of the data and θ with the Gamma prior is

$$p(\underline{x}; \theta)\lambda(\theta) \propto \theta^n \exp(-\theta \sum_1^n x_i)\theta^{\alpha-1} \exp(-\beta\theta) = \theta^{\alpha+n-1} \exp(-(\beta + \sum_1^n x_i)\theta),$$

so it follows that $(\theta|\underline{X}) \sim \text{Gamma}(\alpha + n, \beta + \sum_1^n X_i)$.
(c) The Bayes estimator of θ is

$$\begin{aligned} E(\theta|\underline{X}) &= (\alpha + n)/(\beta + \sum_1^n X_i) \\ &= \frac{1 + \alpha/n}{\overline{X} + \beta/n} \to_{a.s.} \frac{1 + 0}{1/\theta + 0} = \theta_0 \end{aligned}$$

(d) Now

$$
\begin{aligned}
\sqrt{n}(\theta - E(\theta|\underline{X})) &\stackrel{d}{=} \sqrt{n}\left(\text{Gamma}(\alpha+n, \beta+\sum_1^n X_i) - \frac{\alpha+n}{\beta+\sum_1^n X_i}\right) \\
&\stackrel{d}{=} \sqrt{n}\frac{1}{\beta+\sum_1^n X_i}\left(Y_0+Y_1+\cdots+Y_n-(\alpha+n)\right) \\
&= \frac{n}{\beta+\sum_1^n X_i}\frac{1}{\sqrt{n}}((Y_0-\alpha)+(Y_1-1)+\cdots+(Y_n-1)) \\
&\to_d \frac{1}{1/\theta_0}Z \sim N(0,\theta_0^2) = N(0, I^{-1}(\theta_0) \qquad \text{a.s.}
\end{aligned}
$$

where the X_i are i.i.d. Exponential(θ_0). Here $Y_0 \sim$ Gamma$(\alpha, 1)$ is independent of $Y_1, \ldots, Y_n$ which are i.i.d. Gamma$(1,1)$ random variables.

7. (36 points) Suppose that $X_1, \ldots, X_m$ are i.i.d. exponential(μ) and that $Y_1, \ldots, Y_n$ are i.i.d. exponential (ν) and independent of the X_i's. (Thus $P(X_i > x) = \exp(-\mu x)$ for $x \geq 0$ and $P(Y_j > y) = \exp(-\nu y)$ for $y \geq 0$.)

   (a) Find a conditional test of $H: \ \mu \leq \nu$ versus $K: \ \mu > \nu$ which is sufficient for the parameter of interest θ say, and a statistic T which is sufficient for the boundary Θ_B between the null and alternative hypotheses.

   (b) Find a function $V = h(U,T)$ which is monotone increasing in U for each fixed value of $T = t$ and so that V is ancillary on $\Theta_B = \{(\mu,\mu): \ \mu > 0\}$. What do you conclude about V and T under Θ_B?

   (c) Show how to use the result of (b) to carry out the UMP unbiased test of H versus K unconditionally. Identify the relevant distribution explicitly.

   (d) Describe the Bayes test of H versus K (for $0-1$ loss assuming a prior distribution Λ of (μ,ν). Can you describe the rejection region of this test when Λ is the product of two independent Γ priors (i.e. $\mu \sim \Gamma(\alpha,\beta)$ and $\nu \sim \Gamma(\gamma,\delta)$)?

Solution: (a) The joint density of the observations can be written as

$$
\begin{aligned}
p(\underline{x}, \underline{y}; \mu, \nu) &= \mu^m \nu^n \exp(-\mu \sum_{i=1}^m x_i - \nu \sum_{j=1}^n y_j) \\
&= c(\mu,\nu)\exp((\mu-\nu)\sum_{j=1}^n y_j - \mu(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j)) \\
&= c(\mu,\nu)\exp(\theta U + \xi T)
\end{aligned}
$$

where $\theta = \mu - \nu$, $U = \sum_{j=1}^n y_j$, $\xi = -\mu$, and $T \equiv \sum_{i=1}^n x_i + \sum_{j=1}^n y_j$. This is an exponential family, and T is sufficient and (boundedly) complete for the boundary $\Theta_B = \{(\mu, \mu) : \ \mu > 0\}$. Hence a UMP unbiased test is given conditionally on $T = t$ by

$$\phi(\underline{X}, \underline{Y} = \begin{cases} 1 & \text{if } U > c(t) \\ \gamma(t) & \text{if } U = c(t) \\ 0 & \text{if } U < c(t). \end{cases}$$

where $\gamma(t)$ and $c(t)$ are determined so that $E_{\theta=0}\{\phi(\underline{X}, \underline{Y})|T = t\} = \alpha$.
(b) There are several possibilities here. One of them is to note that $V \equiv h(U, T) = U/(T - U) = \sum_{j=1}^n Y_j / \sum_{i=1}^m X_i$ is a monotone function of U for each fixed $T = t$. Hence an equivalent form of the conditional test is as above but with U replaced by V (and different appropriate choices of c and γ). But since $\mu X \sim$ exponential(1) if $X \sim$ exponential (μ), it follows that $\mu \sum_{i=1}^m X_i \sim \text{Gamma}(m, 1)$, and $(\mu/2) \sum_{i=1}^m X_i \sim \text{Gamma}(2m/2, 1/2) = \chi^2_{2m}$. Similarly $\nu \sum_{j=1}^n Y_j \sim \text{Gamma}(n, 1)$, and $(\nu/2) \sum_{j=1}^n Y_j \sim \text{Gamma}(2n/2, 1/2) = \chi^2_{2n}$, and is independent of $\sum X_i$. Hence

$$V = \frac{\sum_{j=1}^n Y_j}{\sum_{i=1}^m X_i} \stackrel{d}{=} \frac{\mu}{\nu} \frac{\chi^2_{2n}}{\chi^2_{2m}} = \frac{\mu}{\nu} F_{2n,2m}$$

and it follows that on the boundary $\mu = \nu$ the distribution of V does not depend on $\mu = \nu$ (it is $F_{2n,2m}$ for all $\mu = \nu$). Hence V is ancillary on Θ_B, and by Basu's theorem it is therefore independent of the complete sufficient statistic T. Another way to do this is to take $V = U/T = U/(U + V)$ with $V \equiv \sum X_i$ which is also clearly monotone in U for each fixed $T = t$. Then since $\nu U \sim \text{Gamma}(n, 1)$ and $\mu V \sim \text{Gamma}(m, 1)$, on the boundary we have $V \sim \text{Beta}(n, m)$ for all $\mu = \nu$. Again V is ancillary and hence independent of T by Basu's theorem.
(c) It follows that the conditional test in B based on V and T reduces to an unconditional test of the form "reject H if $V > F_{2n,2m;\alpha}$ where $P(F_{2n,2m} > F_{2n,2m;\alpha}) = \alpha$.
(d) When (μ, ν) have the prior distribution Λ, the Bayes test of H versus K is "reject H if $P(\mu > \nu|\underline{X}, \underline{Y}) > P(\mu \le \nu|\underline{X}, \underline{Y})$", or, equivalently, if $P(\mu > \nu|\underline{X}, \underline{Y}) > 1/2$. When Λ has density $\lambda(\mu, \nu)$ given by the product of two Gamma densities as described, then the posterior probability in question becomes the probability that an independent $\text{Gamma}(\alpha + m, \beta + \sum_1^m X_i)$ random variable is bigger than a $\text{Gamma}(\gamma + n, \delta + \sum_1^n Y_j)$ random variable, and by scaling this is equivalent to the probability that

$$\frac{1}{\beta + \sum_1^m X_i} \text{Gamma}(\alpha + m, 1) > \frac{1}{\delta + \sum_{j=1}^n Y_j} \text{Gamma}(\gamma + n, 1)$$

and hence the test becomes "reject if

$$P\left(\frac{\delta + \sum_1^n Y_j}{\beta + \sum_1^m X_i} > \frac{\text{Gamma}(\gamma + n, 1)}{\text{Gamma}(\alpha + m, 1)} | \underline{X}, \underline{y}\right) > \frac{1}{2}."$$