Statistics 582, Problem Set 8

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Reading: Chapter 6, sections 6.1 and 6.2 (through page 19)
Due: Wednesday, March 1, 2006.

1. Continuation of problem 1, problem set 7: For P_0 and P_1 as given in problem 1 of problem set # 7, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1) = \int \sqrt{p_0 p_1} d\mu$. For the product laws P_{0n} and P_{1n} (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. P_0 or P_1 respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20,\ 50,\ 100$. What does this imply about the test, ϕ_n say, based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

2. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
   A. Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
   B. What happens to the risk body as n grows or as $\sigma_0 \to 0$?
   C. What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?

3. Let $X_1, \ldots, X_n$ be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
   (a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.
   (b) Find the subclass of the tests that are independent of θ_1. These are UMP tests of H_0 versus $H_1' : \theta > \theta_0$.
   (c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is UMP of size α for testing $H_0' : \theta \le \theta_0$ versus $H_1' : \theta > \theta_0$ but that ϕ is not admissible.
   (d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \le b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \ne \theta_0$.

4. Show that the Cauchy scale family of distributions given by

$$p_\theta(x) = \frac{1}{\pi\theta} \frac{1}{1 + (x/\theta)^2}$$

   does not have monotone likelihood ratio, but that the distribution of the sufficient statistic $|X|$ (where $X \sim P_\theta$ with Cauchy density p_θ as in the last display) does have monotone likelihood ratio.

5. **Optional bonus problem:** (i) Show that the logistic distribution with location parameter θ having density

$$p_\theta(x) = \frac{\exp(x-\theta)}{(1+\exp(x-\theta))^2} = \frac{1}{2(1+\cosh(x-\theta))}$$

has monotone likelihood ratio.
(ii) Unfortunately the result of (i) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ as in (i), then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?