

## Statistics 582, Problem Set 5

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**Reading:** Chapter 5, sections 6-8

**Due:** Wednesday, February 8, 2006.

**Reminder:** Midterm Exam: Friday, February 10.

1. Let  $X$  be a random variable with finite first moment:  $E|X| < \infty$ . Show that  $f(b) \equiv E|X - b|$  is minimized by  $b =$  any median of the distribution  $F$  of  $X$ . [A median  $m$  of  $F$  is any value satisfying  $F(m) = P(X \leq m) \geq 1/2$  and  $1 - F(m-) = P(X \geq m) \geq 1/2$ ; See Lehmann and Casella, TPE, page 62, problems 1.7 and 1.8.
2. Suppose that  $X_1, \dots, X_n$  are i.i.d.  $\text{Exponential}(\theta)$  (so the  $X$ 's have density  $p_\theta(x) = \theta e^{-\theta x} 1_{(0, \infty)}(x)$ . with respect to Lebesgue measure on  $R$ , and that  $\theta \sim \Gamma(\alpha, \beta)$ :

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0, \infty)}(\theta).$$

A. Find the Bayes rule  $d_B(\underline{X})$  for estimation of  $\theta$  with squared error loss  $L(\theta, a) = |\theta - a|^2$ . Find the Bayes rule  $d_{Bw}(\underline{X})$  for estimation of  $\theta$  with weighted squared error loss  $L(\theta, a) = (\theta - a)^2/\theta$ . Is the maximum likelihood estimator among either of these families of Bayes estimators?

B. Are the Bayes estimators  $d_B$  and  $d_{Bw}$  consistent? What are the limit distributions of  $d_B$  and  $d_{Bw}$ ? Compare them with the maximum likelihood estimator.

C. Suppose that instead of the Gamma prior distribution,  $\theta$  has the Pareto( $\theta_0, \alpha$ ) distribution with density  $\lambda$  given by

$$\lambda(\theta) = \left(\frac{\alpha}{\theta_0}\right) \left(\frac{\theta_0}{\theta}\right)^{\alpha+1} 1_{(\theta_0, \infty)}(\theta);$$

here  $E(\theta) = \frac{\alpha}{\alpha-1}\theta_0$  where  $\alpha > 1$  and  $\theta_0 > 0$  are known. What can you say about the Bayes estimator for squared error loss with this prior? For what values of  $\theta_0$  is the Bayes rule consistent?

3. Let  $\Theta = (0, \infty)$ ,  $\mathbf{A} = [0, \infty)$ , let  $X$  have the discrete distribution

$$p(x, \theta) = \binom{r+x-1}{x} \theta^x (\theta+1)^{-(r+x)}, \quad x = 0, 1, 2, \dots$$

where  $r$  is some known positive integer; this is the negative binomial distribution reparametrized so that  $E_\theta X = r\theta$ . Suppose that

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta(\theta + 1)}.$$

- (a) Show that the usual estimator,  $d_0(X) = X/r$  is an equalizer rule.  
 (b) Show that the usual estimator  $d_0$  is generalized Bayes with respect to Lebesgue measure on  $(0, \infty)$  provided  $r > 1$ . (What happens if  $r = 1$ ?)  
 (c) Find Bayes decision rules with respect to the prior distributions  $\Lambda_{\alpha, \beta}$  with densities

$$\lambda_{\alpha, \beta}(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (\theta + 1)^{-(\alpha+\beta)} 1_{(0, \infty)}(\theta),$$

the distribution of  $\theta = Z/(1 - Z)$  where  $Z \sim \text{Beta}(\alpha, \beta)$ .

- (d) Show that  $d(X) = X/(r + 1)$  is minimax. [Note that  $d_0$  is not minimax, hence not admissible.]

4. A. Let  $(X|\sigma^2) \sim N(0, \sigma^2)$ . Show that the conjugate prior for  $\sigma^2$  is the distribution of  $1/Y$  where  $Y$  has a gamma distribution.  
 B. Suppose that  $(X|\theta, \kappa) \sim N(\theta, 1/\kappa)$ ,  $(\theta|\kappa) \sim N(\mu, \tau/\kappa)$ , and  $\kappa \sim \text{Gamma}(\alpha, \beta)$ . Show that the posterior distribution of  $(\theta, \kappa)$  has the same form as the prior.  
 C. Find the marginal posterior distribution for  $\theta$  in B.  
 D. If  $X_1, \dots, X_n$  are i.i.d. as  $X$  in A, find the limiting distribution of the Bayes estimator of  $\theta$ .

5. **Optional bonus problem:** Let  $\Theta = \mathbf{A} = \mathbb{R}$  and let the loss function be

$$L(\theta, a) = \begin{cases} 0, & \text{if } |\theta - a| \leq c, \\ 1, & \text{if } |\theta - a| > c, \end{cases}$$

where  $c > 0$ . The function  $f(b) = EL(Z, b)$  for an arbitrary random variable  $Z$  is minimized when  $b$  is the midpoint of the modal interval of length  $2c$ . Define a “modal interval of length  $2c$ ” so that this makes sense and state a rule for finding Bayes rules using this loss function. [From Ferguson *Mathematical Statistics*, page 51.]