Statistics 582, Problem Set 4

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Reading: Chapter 5, sections 1-6.
Due: Wednesday, February 1, 2006.

1. Let $\Theta = \{0, 1\} = \mathbf{A}$ where 0 = a patient has tuberculosis, 1 = a patient does not have tuberculosis. Let X be the number of positive reactions to two different tuberculosis tests, so that $\mathbf{X} = \{0, 1, 2\}$, and suppose that X has the following distributions

x	0	1	2
$p_0(x)$	.02	.13	.85
$p_1(x)$	.70	.27	.03

If the losses are given by $L(1,1) = L(0,0) = 0$ $L(0,1) = 100$, $L(1,0) = 10$, and the prior $\lambda = (\lambda_0, \lambda_1) = (.2, .8)$, find the Bayes rule d_B and the minimax rule d_M. Plot the risk set and label the non-randomized decision rules.

2. Suppose that $X_n \equiv X \sim \text{Multinomial}_k(n, \underline{\theta})$.
A. Suppose that the prior distribution on θ is given by a Dirichlet distribution, Dirichlet($\underline{\alpha}$):

$$\lambda(\underline{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} 1_{[\underline{\theta}:\ \sum \theta_i = 1]} .$$

Verify the computation of the Bayes estimator for squared error loss given in example 4.3.4
B. What is the posterior distribution for θ? Find the mode of the posterior distribution (along the lines of our computations of the MLE of the multinomial) and compare it with the MLE.
C. Find a minimax estimator d_M of $\underline{\theta}$.

3. Find the limit distribution of the minimax estimator d_M in problem 2 (i.e. $\sqrt{n}(d_M(X_n) - p) \rightarrow_d ??$).

4. Consider testing the simple hypothesis $H_0 :\ X \sim P_0$ versus the simple alternative $H_1 :\ X \sim P_1$. Let ϕ be a test of H_0 versus H_1, and let $a \equiv E_1(1 - \phi)$, $b \equiv E_0 \phi$.
A. Find a test ϕ which minimizes $a + Db$ where D is a fixed number.
B. When $D = 1$, relate the minimized total $a + b$ to the risk and to the total variation distance $d_{TV}(P_0, P_1)$ between P_0 and P_1 (or $\int p_0 \wedge p_1 d\mu$ for a dominating measure μ, e.g. $P_0 + P_1$).

C. Carry the computations of B through in the context of problem 1 when the losses are $L(1,1) = L(2,2) = 0$, $L(1,2) = 10 = L(2,1)$, and the prior distribution is $\lambda = (\lambda_0, \lambda_1) = (.5, .5)$.

5. **Optional bonus problem:** Suppose that $\Theta = \{\theta_1, \theta_2\}$, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, and that the loss function $L(\theta, a)$ is given by the following table:

θ/a	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

Further suppose that $P_{\theta_j}(X = 0) = 1$ for $j = 1, 2$.
A. Find the decision risk set $\mathcal{R}$. B. Find the decision rules that are Bayes with respect to the prior distribution $\lambda = (1, 0)$.
C. Show that the rule d_0 for which $R(\theta_1, d_0) = 1$ and $R(\theta_2, d_0) = 1$ is Bayes with respect to $\lambda = (1, 0)$ and also minimax, but that it is not admissible.
D. Relate this example to our theorem about admissibility of Bayes rules.