Statistics 582, Final Exam

Wellner; 3/13/2006

Instructions: This is an "in class" and "closed-book" exam. Please do it completely on your own with no books or notes.

1. (30 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.

   (a) A uniformly most powerful level α test.

   (b) An unbiased test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (c) A similar test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.

   (d) A level α permutation test.

   (e) The *risk function* of a decision rule d in a decision problem with finite parameter space, action space, sample space, and loss function $L(\theta, a)$.

2. (30 points) **State** *any three* of the following results:

   (a) A theorem relating similar tests to tests with Neyman structure.

   (b) The Wald-Wolfowitz-Noether-Hájek finite sampling central limit theorem.

   (c) A theorem about admissibility properties of the sample mean $\overline{X}$ when sampling from a normal distribution on $\mathbb{R}$ and a contrasting theorem for sampling from a normal distribution on $\mathbb{R}^d$.

   (d) Stein's identity for $Eg'(X)$ where $X \sim N_1(0, \sigma^2)$ and $E|g'(X)| < \infty$.

   (e) A theorem relating Bayes rules to minimax rules and least favorable prior distributions.

   (e) A conditional limit theorem about the large sample behavior of posterior distributions

3. (40 points) **State** and **prove** the Neyman - Pearson lemma.

4. (48 points) Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. Consider testing $H: \; p_2 \le p_1$ versus $K: \; p_2 > p_1$.

   (a) Write the joint density (probability mass function) $p(x, y; p_1, p_2) \equiv P_{p_1,p_2}(X = x, Y = y)$ in exponential family form $c(\theta, \xi) \exp(\theta U(x, y) + \xi T(x, y)) h(x, y)$ where $T(x, y)$ is sufficient for the boundary $\Theta_B = \{(p_1, p_2) \in [0, 1]^2 : \; p_1 = p_2\}$ and $U(x, y) = y$.

(b) Show carefully that testing H versus K is equivalent to testing $H_1 : \theta \leq 0$ ($\xi =$ anything) versus $K_1 : \theta > 0$ ($\xi =$ anything).

(c) What is the conditional distribution of $U(X,Y) = Y$ given $T = t$ under $p_1 = p_2$? Compute it explicitly and give its name.

(d) Find the UMP unbiased test of size α as explicitly as possible when $m = 3$, $n = 2$, $t = 2$, and $\alpha = 1/10$.

(e) Relate the conditional distribution of the test statistic involved in (c) to a problem involving sampling without replacement from a finite population. Identify the contents of the urn (i.e. the numbers on the balls in the urn) and calculate the mean and variance of Y given $T = t$ in this conditional distribution.

(f) Use the results of (e) together with the Wald-Wolfowitz-Noether-Hájek CLT to show a conditional (on $T = t$) CLT for Y appropriately centered and normalized if $0 < \liminf(m/N) \leq \limsup(m/N) < 1$. [Make sure that you verify the key hypothesis of the theorem.]

5. (36 points) A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.1	.05	.35	.5
$p_1(x)$	.3	.25	.35	.1

(a) For the usual $0 - 1$ loss, find a most powerful test of size .10 for testing $H : p = p_0$ versus $K : p = p_1$ and determine its power.

(b) Find a test ϕ which minimizes the sum of risks $E_0\phi + E_1(1 - \phi)$.

(c) If the losses are $L(1,1) = L(0,0) = 0$, $L(0,1) = 10$, $L(1,0) = 5$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.7, .3)$, find the Bayes rule d_B and the minimax rule d_M.

Do either problem 6 or problem 7.

6. (36 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. exponential(θ) random variables so that $1 - F_\theta(x) = \exp(-\theta x)$ for $x \geq 0$ and $\theta > 0$.
(a) Find the MLE of θ.
(b) If $\theta \sim \text{Gamma}(\alpha, \beta)$ so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0,\infty)}(\theta)$$

and $E(\theta) = \alpha/\beta$, find the posterior distribution of θ.

(c) Find the Bayes estimator of θ for squared error loss. Is it consistent?

(d) What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?

7. (36 points) Suppose that $X_1, \ldots, X_m$ are i.i.d. exponential(μ) and that $Y_1, \ldots, Y_n$ are i.i.d. exponential (ν) and independent of the X_i's. (Thus $P(X_i > x) = \exp(-\mu x)$ for $x \geq 0$ and $P(Y_j > y) = \exp(-\nu y)$ for $y \geq 0$.)

(a) Find a conditional test of $H: \ \mu \leq \nu$ versus $K: \ \mu > \nu$ which is sufficient for the parameter of interest θ say, and a statistic T which is sufficient for the boundary Θ_B between the null and alternative hypotheses.

(b) Find a function $V = h(U, T)$ which is monotone increasing in U for each fixed value of $T = t$ and so that V is ancillary on $\Theta_B = \{(\mu, \mu) : \ \mu > 0\}$. What do you conclude about V and T under Θ_B?

(c) Show how to use the result of (b) to carry out the UMP unbiased test of H versus K unconditionally. Identify the relevant distribution explicitly.

(d) Describe the Bayes test of H versus K (for $0-1$ loss assuming a prior distribution Λ of (μ, ν). Can you describe the rejection region of this test when Λ is the product of two independent Γ priors (i.e. $\mu \sim \Gamma(\alpha, \beta)$ and $\nu \sim \Gamma(\gamma, \delta)$)?