Statistics 582, Handout
Existence of Consistent Roots of Estimating Equations
Wellner; 12/10/97 - 1/5/98

First we need the statement of Brouwer's fixed point theorem. Let $S_\delta(\theta_0) = \{\theta \in \Theta \subset R^d : \ |\theta - \theta_0| \le \delta\}$.

Theorem. (Brouwer). If $f : S_\delta(\theta_0) \to S_\delta(\theta_0)$ is a continuous map, there exists at least one $\theta^* \in S_\delta(\theta_0)$ such that $f(\theta^*) = \theta^*$.

For Brouwer's theorem, see Dunford and Schwartz (1957), Part I, page 453, or Ziedler (1986), Part I, page 51, Part IV, page 799.

This leads to a lemma concerning the existence of zeros which is due to Atchison and Silvey (1958).

Lemma. If $g : S_\delta(\theta_0) \to R^d$ is continuous and if $(\theta - \theta_0)^T g(\theta) < 0$ for all $\theta \in \partial S_\delta(\theta_0) = \{\theta : \ |\theta - \theta_0| = \delta\}$, then there exists a point $\hat{\theta} \in S_\delta^o(\theta_0) \equiv \{\theta : |\theta - \theta_0| < \delta\}$ and $g(\hat{\theta}) = 0$.

Proof. Clearly g cannot have roots on the boundary of $S_\delta(\theta_0)$ because $(\theta - \theta_0)^T g(\theta) < 0$ on the boundary by hypothesis. Assume $g(\theta) \ne 0$ for all $\theta \in S_\delta^o(\theta_0)$ and hence $g(\theta) \ne 0$ on $S_\delta(\theta)$ by the preceding argument. Define $f : S_\delta(0) \to S_\delta(0)$ by

$$f(x) = \delta \frac{g(\theta_0 + x)}{|g(\theta_0 + x)|}.$$

Then f is continuous and $|f(x)| = \delta$ on $S_\delta(0)$. By Brouwer's fixed point theorem there exists $x^* \in S_\delta(0)$ such that $f(x^*) = x^*$. Hence $|x^*| = |f(x^*)| = \delta > 0$. Let $\hat{\theta} = \theta_0 + x^*$. Then $|\hat{\theta} - \theta_0| = \delta$ and

$$\begin{aligned}
(\hat{\theta} - \theta_0)^T g(\hat{\theta}) &= (\hat{\theta} - \theta_0)^T g(\theta_0 + x^*) = x^{*T} g(\theta_0 + x^*) \\
&= x^{*T} f(x^*) \frac{|g(\theta_0 + x^*)|}{\delta} \\
&= |x^*|^2 \frac{|g(\theta_0 + x^*)|}{\delta} > 0 \,,
\end{aligned}$$

which contradicts $(\theta - \theta_0)^T g(\theta) < 0$ for all $|\theta - \theta_0| = \delta$. Hence there exists $\hat{\theta} \in S_\delta^o(\theta_0)$ satisfying $g(\hat{\theta}) = 0$.

$\square$

Theorem: Suppose that A0 - A4 hold. Then there exists a set $M_n \subset \mathcal{X}^n$ such that $P_{\theta_0}(M_n) \to 1$ and such that for $\underline{X} \in M_n$, the score equations $\dot{\underline{l}}_n(\theta|\underline{X}) = \underline{0}$ have a root $\tilde{\theta}_n$ satisfying $\tilde{\theta}_n 1_{M_n} \to_p \theta_0$. Defining $\tilde{\theta}_n$ arbitrarily outside of M_n yields

$$\tilde{\theta}_n = \tilde{\theta}_n 1_{M_n} + \tilde{\theta}_n 1_{M_n^c} \to_p \theta_0 \,.$$

Proof. This proof is closely related to the proof of theorem 4.1.5 part (i), but now we make use of the above lemma. We first write

$$\begin{aligned} \frac{1}{n}\dot{\underline{l}}_n(\theta) &= \frac{1}{n}\dot{\underline{l}}_n(\theta_0) + \frac{1}{n}\ddot{\underline{l}}_n(\theta_0)(\theta-\theta_0) \\ &\qquad + \int_0^1 \left\{ \frac{1}{n}\ddot{\underline{l}}_n(\theta_0 + t(\theta-\theta_0)) - \frac{1}{n}\ddot{\underline{l}}_n(\theta_0) \right\} (\theta-\theta_0)\, dt. \end{aligned}$$

We define g of the lemma by $g(\theta) = n^{-1}\dot{\underline{l}}_n(\theta)$; note that continuity of g folows from hypthosesis A3. By our proof of part (i) of Theorem 4.1.5 we see that there is a set M_n with $P_0(M_n) \to 1$ such that for $\underline{X} \in M_n$ we have

$$(\theta-\theta_0)\frac{1}{n}\dot{\underline{l}}_n(\theta) < -\delta^2 \frac{C}{2A} < 0$$

for all $\theta \in \partial S_\delta(\theta_0) = \{\theta : |\theta - \theta_0| = \delta\}$ if $\delta < C/2A$ with $C \equiv \lambda_d/2$ and $A \equiv 1 + \sum_{j,k,l} m_{j,k,l}$. Thus on the sets M_n the hypotheses of the Lemma are satisfied and there exists a root $\tilde{\theta}_n$ of the score equations $\dot{\underline{l}}_n(\theta|\underline{X}) = \underline{0}$.

$\square$

References:

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