STATISTICS 581:
Solutions, Day 1 Quiz, Fall 2002

1. **State** the (Lindeberg) Central Limit Theorem.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$, then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d N(0, \sigma^2)\,.$$

In other words, if $Z \sim N(0,1)$ and Φ denotes the distribution function of a standard normal random variable,

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy\,,$$

then, for all $t \in R$,

$$P(\sqrt{n}(\overline{X}_n - \mu) \le t) \to P(\sigma Z \le t) = \Phi(t/\sigma)\,. \tag{1}$$

[Note: the hypothesis $\sigma^2 < \infty$ is also necessary: if i.i.d. random variables X_i satisfy (1) then $\sigma^2 < \infty$.]

2. Suppose that $U \sim$ Uniform$(0,1)$. For what values of $c \in R$ is it true that $E(U^c) < \infty$? For the values of c for which the integral is finite, compute it explicitly.

Solution: We first compute for $c > -1$. In this case $c + 1 > 0$ and

$$\begin{aligned} E(U^c) &= \int_0^1 u^c du = \frac{1}{c+1} u^{c+1}\Big|_0^1 \\ &= \frac{1}{c+1}\left\{1^{c+1} - 0^{c+1}\right\} = \frac{1}{c+1}\,. \end{aligned}$$

Note that $\lim_{c \downarrow -1} E(U^c) = \infty$. When $c = -1$,

$$E(U^{-1}) = \int_0^1 \frac{1}{u} du = \log(u)\Big|_0^1 = \log(1) - \log(0) = \infty\,.$$

When $c < -1$, then $c + 1 < 0$ and hence

$$E(U^c) = \int_0^1 u^c du = \frac{1}{c+1} u^{c+1}\Big|_0^1 = \frac{-1}{c+1}\left\{\frac{1}{0^{-(c+1)}} - \frac{1}{1^{-(c+1)}}\right\} = \infty\,.$$

It follows that $E(U^c) < \infty$ if and only if $c > -1$.

3. Suppose that $U \sim$ Uniform$(0,1)$ and $V_n = j/n$ on the set $[(j-1)/n \le U < j/n]$ for $j = 1, \ldots, n$. What is the distribution of V_n?

Solution: Note that $P(V_n = j/n) = P((j-1)/n \le U < j/n) = 1/n$ for $j = 1, \ldots, n$. Thus V_n has a discrete uniform distribution on the set $\{1/n, \ldots, j/n, \ldots, n/n\}$.

4. **Define** what is meant by:

A. V_n converges in distribution to V for random variables V and V_n, $n \ge 1$.

B. V_n converges in probability to V.

C. Does the sequence V_n defined in problem 3 converge in distribution? Does it converge in probability?

Solution: A. V_n converges in distribution to V if

$$F_n(x) = P(V_n \le x) \to P(V \le x) = F(x)$$

for all $x \in C_F = \{x \in R : \ F \text{ is continuous at } x\}$.
B. V_n converges in probability to V if

$$P(|V_n - V| > \epsilon) \to 0 \qquad \text{as } \ n \to \infty \ \text{ for every } \ \epsilon > 0\,.$$

C. The sequence of random variables defined in problem 3 converges both in distribution and in probability to $U \sim$ Uniform$(0,1)$. To see the convergence in distribution, note that the distribution function F_n of V_n is given by

$$F_n(x) = P(V_n \le x) = \sum_{j=1}^{n} \frac{j-1}{n} 1_{[(j-1)/n, j/n)}(x) + 1_{[1,\infty)}(x)\,,$$

and we have, for the Uniform$(0,1)$ distribution function F of U, $|F_n(x) - F(x)| \le 1/n$ for all x. Hence $F_n(x) \to F(x)$ for all x; i.e. $V_n \to_d U$. To see that $V_n \to_p U$, note that

$$P(|V_n - U| > \epsilon) = \begin{cases} n(1/n - \epsilon) = 1 - n\epsilon\,, & \text{if } 1/n > \epsilon \\ 0\,, & \text{if } 1/n \le \epsilon\,. \end{cases}$$

Hence it follows that $V_n \to_p U$.

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \ge 1$ define $X_n = n1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Compute $E(X_n)$. Does it converge to $E(X')$ for some X' (possibly different than the X in (a) and (b)) ?

Solution: Here it is easier to answer (b) first, then (a):
(b) Note that for any $\epsilon \in (0,1)$

$$P(|X_n| > \epsilon) = P(U \le 1/n) = 1/n \to 0\,.$$

Hence $X_n \to_p 0$. (a) Since convergence in probability implies convergence in distribution, the result of (b) implies that $X_n \to_d 0$. [In fact, $X_n \to_{a.s.} 0$: for $U(\omega) \in (0,1]$ we have $1/n < U(\omega)$ for all $n \ge N(\omega)$ sufficiently large, and hence $X_n(\omega) = 0$ for $n \ge N(\omega)$. But $P(U \in (0,1]) = 1$, so it follows that $X_n \to_{a.s.} 0$. Since $\to_{a.s.}$ implies $\to_p$, this also yields the desired conclusion(s).]
(c) The expectation is

$$E(X_n) = E\{n1_{[0,1/n]}(U)\} = nP(0 \le U \le 1/n) = n(1/n) = 1$$

for all n. This gives an example for which $\lim_n E(X_n) \ne E(\lim_n X_n)$. But note that $X_n \ge 0$ and Fatou's lemma (Theorem 0.2.2, page 9, Chapter 0 notes) does indeed hold with strict inequality:

$$0 = E(0) = E(\underline{\lim} X_n) < \underline{\lim} E(X_n) = \underline{\lim} 1 = 1\,.$$

6. Suppose that $X_1, \ldots, X_n, \ldots$ are independent and identically distributed Bernoulli(p) random variables (i.e. $P(X_i = 1) = p = 1 - P(X_i = 0)$ for $i = 1, 2, \ldots$). Let $T_n = X_1 + \ldots + X_n$.
(a) What is the distribution of T_n?
(b) Does $\overline{X}_n = n^{-1} T_n \rightarrow_p$ something? If so, what is "something"?
(c) Does $\sqrt{n}(\overline{X}_n - p) \rightarrow_d$ something? If so, what is "something"?
(d) What is the Cramér-Rao bound for unbiased estimators of p?

Solution: (a) $T_n \sim$ Binomial(n, p).
(b) $\overline{X}_n = n^{-1} T_n \rightarrow_p p$ by the weak law of large numbers.
(c) $\sqrt{n}(\overline{X}_n - p) \rightarrow_d N(0, p(1-p))$ by the Central Limit Theorem.
(d) The density (mass function) for a sample of size one is

$$p(x; p) = p^x (1-p)^{1-x} \quad \text{for} \quad x \in \{0, 1\}\,.$$

Thus

$$logp(x; p) = x \log p + (1 - x) \log(1 - p)$$

and the score fucntion for estimating p is given by

$$\dot{l}_p(x) = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}\,.$$

Thus the information for p in a sample of size $n = 1$ is

$$I(p) = E(\dot{l}_p(X)^2) = \frac{E(X-p)^2}{[p(1-p)]^2} = \frac{1}{p(1-p)}\,.$$

(Alternatively,

$$I(p) = -E(\ddot{l}_{pp}(X)) = E\left\{\frac{X}{p^2} + \frac{1-X}{(1-p)^2}\right\} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}\,.)$$

It follows that the Cramér-Rao lower bound for unbiased estimates of p based on n observations is:

$$\frac{1}{nI(p)} = \frac{p(1-p)}{n}\,.$$