

## Statistics 581, Problem Set 9 Solutions

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1. (a) Lehmann and Casella, problem 6.3.1, page 501: Let  $X$  have the binomial distribution  $Bin(n, p)$ ,  $0 \leq p \leq 1$ . Determine the MLE of  $p$ :
  - (i) by the usual calculus method determining the maximum of a function.
  - (ii) by showing that  $p^x q^{n-x} \leq (x/n)^x [(n-x)/n]^{n-x}$ .
- (b) Lehmann and Casella, problem 6.3.2, page 501: In the preceding problem, show that the MLE does not exist when  $p$  is restricted to  $0 < p < 1$  and when  $X = 0$  or  $X = 1$ .
- (c) Lehmann and Casella, problem 6.3.5, page 501: Here is a (slight) rephrasing of the problem: Let  $X \sim \text{Bernoulli}(p)$ :  $P(X = 0) = 1 - p = q$ ,  $P(X = 1) = p$ . Suppose that it is known that  $1/3 \leq p \leq 2/3$ .
  - (i) Find the MLE;
  - (ii) Show that the expected squared error of the MLE is uniformly larger than that of  $\delta(X) = 1/2$ .

**Solution:** (a)(i) Since  $\log P_p(X = x) = x \log p + (n-x) \log(1-p)$ , we have  $l(p|X) = X \log p + (n-X) \log(1-p)$ ; differentiating this with respect to  $p$  yields

$$l'(p|X) = \frac{X}{p} - \frac{n-X}{1-p} = \frac{X(1-p) - (n-X)p}{p(1-p)}$$

and this equals 0 if  $p = \hat{p} \equiv X/n$ . Since the second derivative is

$$l''(p|X) = -\frac{X}{p^2} - \frac{n-X}{(1-p)^2} < 0$$

it follows that  $\hat{p} = X/n$  is the MLE of  $p \in [0, 1]$ .

(a)(ii) Since  $(\prod_{i=1}^n y_i)^{1/n} \leq n^{-1}(y_1 + \dots + y_n)$  for any numbers  $y_i \geq 0$ , it follows, with  $y_i \equiv np/X$  for  $i = 1, \dots, X$ , and  $y_i \equiv nq/(n-X)$ ,  $i = X+1, \dots, n$ , that

$$\left\{ \left( \frac{np}{X} \right)^X \left( \frac{nq}{n-X} \right)^{n-X} \right\}^{1/n} \leq n^{-1} \left\{ X \frac{np}{X} + (n-X) \frac{nq}{n-X} \right\} = 1,$$

or, equivalently,

$$p^X (1-p)^{n-X} \leq \left( \frac{X}{n} \right)^X \left( \frac{n-X}{n} \right)^{n-X},$$

with equality if and only if  $p = X/n \equiv \hat{p}$ . Thus  $\hat{p} = X/n$  is the MLE of  $p \in [0, 1]$ .

(b) When the closed interval  $[0, 1]$  is replaced by the open interval  $(0, 1)$ , then the MLE exists if  $0 < X < n$  and is  $\hat{p} = X/n \in (0, 1)$  in this case. If  $X = 0$ , then the log-likelihood equals  $n \log(1-p)$ , so  $\sup_{p \in (0,1)} l(p) = 0$ , but this supremum is not achieved (in the set  $(0, 1)$ ). Thus the MLE does not exist in this case. Similarly, if  $X = n$ , the the log-likelihood equals  $n \log p$ , so  $\sup_{p \in (0,1)} l(p) = 0$ , but this supremum is not achieved (in the set  $(0, 1)$ ).

(c)(i) For the more general case in which  $X \sim \text{Binomial}(n, p)$  From (a), the MLE  $\hat{p}$  of  $p \in [1/3, 2/3]$  is

$$\hat{p} = \begin{cases} X/n, & \text{if } X/n \in [1/3, 2/3], \\ 1/3, & \text{if } X/n < 1/3, \\ 2/3, & \text{if } X/n > 2/3. \end{cases}$$

For  $n = 1$ , this implies that the MLE is  $1/3$  if  $X = 0$  and  $2/3$  if  $X = 1$ .

(ii) Now the estimator  $\delta(X) = 1/2$  has expected squared error

$$R_1(p) \equiv E_p(\delta(X) - p)^2 = (1/2 - p)^2, \quad 1/3 \leq p \leq 2/3.$$

On the other hand the MLE  $\hat{p}$  has expected squared error

$$\begin{aligned} R_2(p) \equiv E_p(\hat{p} - p)^2 &= p(2/3 - p)^2 + q(1/3 - p)^2 \\ &> (1/2 - p)^2 = R_1(p) \end{aligned}$$

by noting that  $R_2(1/3) = 1/3^3 > 1/6^2 = R_1(1/3)$ , and  $R_2(1/2) = (1/2)(1/6)^2 + (1/2)(1/6)^2 = 1/6^2 > 0 = R_1(1/2)$ . See the following Figure 1 for a comparison of these two mean-square errors.

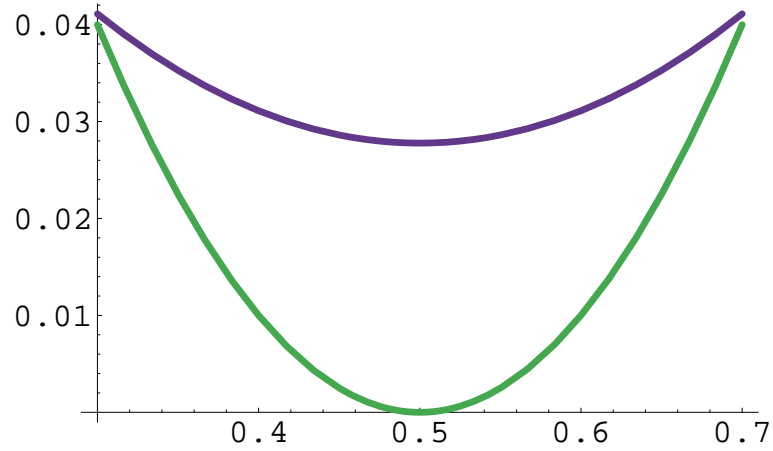


Figure 1: Mean-Square Errors.

(d) First consider problem 3.15(c): Here the log-likelihood for the scale family for a density  $f$  is

$$l_n(a) = n \log a + \sum_{i=1}^n \log f(aX_i), \quad a > 0.$$

Hence the score equation is

$$\begin{aligned} 0 = \dot{l}_n(a) &= \frac{n}{a} + \sum_{i=1}^n \frac{f'(aX_i)}{f(aX_i)} X_i \\ &= \frac{1}{a} \left\{ n - \sum_{i=1}^n g(aX_i) \right\} \equiv \frac{1}{a} \{n - h_n(a)\} \end{aligned}$$

where  $g(x) \equiv -xf'(x)/f(x)$ . If  $xf'(x)/f(x)$  is strictly decreasing in  $x$ , then  $g(x)$  is strictly increasing in  $x$ , and hence  $h_n(a)$  is strictly increasing in  $a$ . Hence there is at most one value of  $a$  satisfying  $h_n(a) = n$ . If a solution exists, it is unique.

For the particular case of a Cauchy density  $f$ , it is easy to compute

$$g(x) = \frac{2x^2}{1+x^2}$$

which is strictly increasing (from 0 at  $x = 0$  to 2 at  $x = \infty$ ). Moreover, in this case the likelihood equation becomes

$$h_n(a) = \sum_{i=1}^n \frac{2a^2 X_i^2}{1+a^2 X_i^2} = n.$$

But the left side converges to 0 as  $a \downarrow 0$ , and converges to  $2n$  as  $a \uparrow \infty$ . Since it is monotone increasing and continuous, there is a unique solution  $\hat{a}_n$ . All the hypotheses of Theorem 1.5 hold in this case, and

$$I_{scale}(f) = \int_{-\infty}^{\infty} \left\{ 1 + x \frac{f'(x)}{f(x)} \right\}^2 f(x) dx = \frac{1}{2}.$$

Thus it follows that

$$\sqrt{n}(\hat{a}_n - a) \rightarrow_d N(0, 2a^2).$$

2. Suppose that  $(Y|Z) \sim \text{Weibull}(\lambda^{-1}e^{-\gamma Z}, \beta)$ , as in problem 2 of problem set # 8, where  $Z \sim \text{Bernoulli}(\eta)$ . Reparametrize as in problem set # 8 so that the joint density is  $p_\theta(y, z)$  with  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$  where  $\theta_4 = \eta$ . Thus the conditional hazard function is

$$\lambda_\theta(t|z) = \theta_3 \theta_2 \exp(\theta_1 z) t^{\theta_3 - 1},$$

$Z \sim \text{Bernoulli}(\eta)$ , and  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$  with  $\theta_4 \equiv \eta$ . Let  $X = (Y, Z)$ , and suppose that we observe  $X_1, \dots, X_n$  i.i.d. as  $X$ .

- Find the score equations for estimation of  $\theta$ .
- Give conditions on the data  $X_1, \dots, X_n = (Y_1, Z_1), \dots, (Y_n, Z_n)$  guaranteeing that the score equations have a unique solution which maximizes the likelihood. Call the resulting estimators  $\hat{\theta}_n$ .
- What does theorem 4.1.2 (Chapter 4, page 5), say about the asymptotic distribution of  $\sqrt{n}(\hat{\theta}_n - \theta_0)$  when the distribution of the data is given by  $P_{\theta_0}$ .
- Suppose that  $\theta_1 \neq \theta_0$  is the “true” value of the parameter  $\theta$ , and we consider the likelihood ratio  $L_n(\theta_1)/L_n(\theta_0)$  where  $L_n(\theta) \equiv \prod_{i=1}^n p_\theta(X_i)$ . Show that  $n^{-1} \log(L_n(\theta_1)/L_n(\theta_0)) \rightarrow_p$  some constant, and identify the constant as explicitly as possible in terms of  $\theta_1, \theta_0$ .

**Solution:** (a) From problem set #8 we know that

$$\begin{aligned} \dot{\mathbf{i}}_1(Y, Z) &= Z(1 - \theta_2 e^{\theta_1 Z} Y^{\theta_3}), \\ \dot{\mathbf{i}}_2(Y, Z) &= \frac{1}{\theta_2} - e^{\theta_1 Z} Y^{\theta_3}, \\ \dot{\mathbf{i}}_3(Y, Z) &= \frac{1}{\theta_3} + \log Y(1 - \theta_2 e^{\theta_1 Z} Y^{\theta_3}), \\ \dot{\mathbf{i}}_4(Y, Z) &= \frac{Z}{\eta} - \frac{1 - Z}{1 - \eta} = \frac{Z - \eta}{\eta(1 - \eta)}. \end{aligned}$$

Hence the score equations for  $\theta = (\theta_1, \dots, \theta_4)$  are

$$0 = \sum_{i=1}^n \dot{\mathbf{i}}_1(Y_i, Z_i) = \sum_{i=1}^n Z_i(1 - \theta_2 e^{\theta_1 Z_i Y_i^{\theta_3}}), \quad (0.1)$$

$$0 = \sum_{i=1}^n \dot{\mathbf{i}}_2(Y_i, Z_i) = \sum_{i=1}^n \left\{ \frac{1}{\theta_2} - e^{\theta_1 Z_i Y_i^{\theta_3}} \right\}, \quad (0.2)$$

$$0 = \sum_{i=1}^n \dot{\mathbf{i}}_3(Y_i, Z_i) = \sum_{i=1}^n \left\{ \frac{1}{\theta_3} + \log Y_i(1 - \theta_2 e^{\theta_1 Z_i Y_i^{\theta_3}}) \right\}, \quad (0.3)$$

$$0 = \sum_{i=1}^n \dot{\mathbf{i}}_4(Y_i, Z_i) = \sum_{i=1}^n \frac{Z_i - \eta}{\eta(1 - \eta)}. \quad (0.4)$$

(b) Let  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_4)$  denote the solution of this system of equations, assuming that it exists. It follows easily that

$$\hat{\theta}_4 = \hat{\eta} = \bar{Z}_n, \quad (0.5)$$

and, from (0.2),

$$\frac{1}{\hat{\theta}_2} = \frac{1}{n} \sum_{i=1}^n e^{\hat{\theta}_1 Z_i Y_i^{\hat{\theta}_3}}. \quad (0.6)$$

Substitution of (0.6) into (0.1) and (0.3) yields

$$\sum_{i=1}^n Z_i \left( 1 - \frac{e^{\hat{\theta}_1 Z_i Y_i^{\hat{\theta}_3}}}{n^{-1} \sum_{j=1}^n e^{\hat{\theta}_1 Z_j Y_j^{\hat{\theta}_3}}} \right) = 0, \quad (0.7)$$

and

$$\frac{n}{\hat{\theta}_3} + \sum_{i=1}^n \log Y_i \left( 1 - \frac{e^{\hat{\theta}_1 Z_i Y_i^{\hat{\theta}_3}}}{n^{-1} \sum_{j=1}^n e^{\hat{\theta}_1 Z_j Y_j^{\hat{\theta}_3}}} \right) = 0. \quad (0.8)$$

Thus we have reduced the original problem to two equations, (0.7) and (0.8) in the two parameters  $\theta_1$  and  $\theta_3$ . Since each  $Z_i$  is either 0 or 1, the first of these two equations can be rewritten as

$$\sum_{i:Z_i=1} \left( 1 - \frac{e^{\hat{\theta}_1 Z_i Y_i^{\hat{\theta}_3}}}{n^{-1} \sum_{j=1}^n e^{\hat{\theta}_1 Z_j Y_j^{\hat{\theta}_3}}} \right) = 0, \quad (0.9)$$

or equivalently

$$\sum_{i:Z_i=1} \left\{ e^{\hat{\theta}_1} \sum_{j:Z_j=1} Y_j^{\hat{\theta}_3} - n e^{\hat{\theta}_1} Y_i^{\hat{\theta}_3} + \sum_{j:Z_j=0} Y_j^{\hat{\theta}_3} \right\} = 0, \quad (0.10)$$

or

$$\begin{aligned} e^{\hat{\theta}_1} \left\{ \left( \sum_1^n Z_i \right) \left( \sum_1^n Z_j Y_j^{\hat{\theta}_3} \right) - n \sum_1^n Z_i Y_i^{\hat{\theta}_3} \right\} \\ + \left( \sum_1^n Z_i \right) \left( \sum_1^n (1 - Z_i) Y_i^{\hat{\theta}_3} \right) = 0. \end{aligned}$$

Assuming that  $0 < \sum_1^n Z_i < n$  (i.e. not all  $Z_i = 1$  and not all  $Z_i = 0$ ), this yields

$$e^{\widehat{\theta}_1} = \frac{\sum_1^n Z_i \sum_1^n (1 - Z_i) Y_i^{\widehat{\theta}_3}}{(n - \sum_1^n Z_i) \sum_1^n Z_i Y_i^{\widehat{\theta}_3}}. \quad (0.11)$$

Note that the right side of (0.11) converges to 1 as  $\widehat{\theta}_3 \rightarrow 0$ , and as  $\widehat{\theta}_3 \rightarrow \infty$  it converges to

$$\frac{(\sum_1^n Z_i)(1 - Z_{(n)})}{(n - \sum_1^n Z_i)Z_{(n)}} = \begin{cases} 0, & \text{if } Z_{(n)} = 1 \\ \infty, & \text{if } Z_{(n)} = 0. \end{cases}$$

Here  $Z_{(n)}$  is the  $Z_j$  corresponding to the largest  $Y_j$ , namely  $Y_{(n)}$ . On the other hand, for fixed values of  $\theta_1$ , the third score equation (0.8) is

$$\frac{n}{\widehat{\theta}_3} + \sum_{i=1}^n \log Y_i \left( 1 - \frac{e^{\theta_1 Z_i Y_i^{\widehat{\theta}_3}}}{n^{-1} \sum_{j=1}^n e^{\theta_1 Z_j Y_j^{\widehat{\theta}_3}}} \right) = 0 \quad (0.12)$$

or equivalently,

$$\frac{n}{\widehat{\theta}_3} + \sum_{i=1}^n \log Y_i \left( 1 - \frac{e^{\theta_1 Z_i Y_i^{\widehat{\theta}_3}}}{n^{-1} \sum_{j=1}^n e^{\theta_1 Z_j Y_j^{\widehat{\theta}_3}}} \right) = 0 \quad (0.13)$$

or,

$$h(\widehat{\theta}_3, \theta_1) \equiv \frac{\sum_{i=1}^n [Z_i e^{\theta_1} + (1 - Z_i)] Y_i^{\widehat{\theta}_3} \log Y_i}{\sum_{j=1}^n [Z_j e^{\theta_1} + (1 - Z_j)] Y_j^{\widehat{\theta}_3}} - \frac{1}{\widehat{\theta}_3} = \frac{1}{n} \sum_1^n \log Y_i \quad (0.14)$$

where

$$h(v, w) \equiv \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v \log Y_i}{\sum_{j=1}^n [Z_j e^w + (1 - Z_j)] Y_j^v} - \frac{1}{w}. \quad (0.15)$$

Now for each fixed  $w$  we have

$$\begin{aligned} \frac{\partial}{\partial v} h(v, w) &= \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v (\log Y_i)^2}{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v} \\ &\quad - \left( \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v \log Y_i}{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v} \right)^2 + \frac{1}{v^2} \\ &\equiv I + II > I \end{aligned}$$

and furthermore

$$I = \sum a_i^2 p_i - \left( \sum a_i p_i \right)^2 = \text{Var}_p(a) \quad (0.16)$$

with  $a_i \equiv \log Y_i$ ,

$$p_i \equiv \frac{[Z_i e^w + (1 - Z_i)] Y_i^v}{\sum [Z_j e^w + (1 - Z_j)] Y_j^v}.$$

Thus  $I > 0$  if not all the  $Y_i$ 's are equal, and hence

$$\frac{\partial}{\partial v} h(v, w) > 0. \quad (0.17)$$

Moreover

$$-\infty = \lim_{v \searrow 0} h(v, w) < \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] \log Y_i}{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)]} < \log Y_{(n)} = \lim_{v \rightarrow \infty} h(v, w).$$

To see this last limit, note that with

$$p^{(i)} \equiv \frac{[Z_{(i)} e^w + (1 - Z_{(i)})] Y_{(i)}^v}{\sum_{j=1}^n [Z_j e^w + (1 - Z_j)] Y_j^v}$$

(here  $Y_{(1)} \leq \dots \leq Y_{(n)}$  are the ordered  $Y_i$ 's and  $Z_{(1)}, \dots, Z_{(n)}$  are the corresponding  $Z$ 's),

$$\begin{aligned} p^{(i)} &= \frac{[Z_{(i)} e^w + (1 - Z_{(i)})]}{\sum_{j=1}^n [Z_{(j)} e^w + (1 - Z_{(j)})] (Y_{(j)}/Y_{(i)})^v} \\ &\rightarrow \begin{cases} 0, & i < n \quad (\text{so } (Y_{(n)}/Y_{(i)}) > 1) \\ 1, & i = n \quad (\text{so } Y_{(j)}/Y_{(n)} < 1, \quad j < n) \end{cases} \end{aligned}$$

as  $v \rightarrow \infty$ . Thus (0.14) has a unique solution  $\widehat{\theta}_3 = \widehat{\theta}_3(\theta_1)$  for each fixed  $\theta_1$ . This is as far as I have succeeded in going with this problem. A summary of what I know is as follows: there is clearly trouble in estimating  $\theta_1$  if all the  $Z_i$ 's are either 0 or 1; and monotonicity of the (condensed version of) the score equation for  $\theta_3$  will fail if all the  $Y_j$ 's are equal. It seems likely that there is a unique solution when  $0 < \sum_1^n Z_i < n$  and  $Y_j \neq Y_k$  for some  $j \neq k$ , but I do not yet have a proof.

(c) Theorem 4.1.2 yields (after somewhat painful verification of the third derivative condition)

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \rightarrow_d N_4(0, I^{-1}(\theta_0))$$

where  $I(\theta_0)$  was computed in problem 8.2.

(d) When  $\theta_1 \neq \theta_0$  is true,

$$\begin{aligned} n^{-1} \log \frac{L_n(\theta_1)}{L_n(\theta_0)} &= n^{-1} \sum_{i=1}^n \log \frac{p_{\theta_1}(Y_i, Z_i)}{p_{\theta_0}(Y_i, Z_i)} \rightarrow_{a.s.} E_{\theta_1} \log \frac{p_{\theta_1}(Y_1, Z_1)}{p_{\theta_0}(Y_1, Z_1)} \\ &= K(P_{\theta_1}, P_{\theta_0}) \\ &= \log \frac{\theta_{12}\theta_{13}}{\theta_{02}\theta_{03}} + (\theta_{11} - \theta_{01})\theta_{14} + (\theta_{13} - \theta_{03})E_{\theta_1} \log Y - E_{\theta_1} W_1 \\ &\quad + E_{\theta_1} W_0 + \theta_{14} \log \frac{\theta_{14}}{\theta_{04}} + (1 - \theta_{14}) \log \frac{1 - \theta_{14}}{1 - \theta_{04}}, \end{aligned}$$

where

$$\begin{aligned} W_1 &\equiv \theta_{12} e^{\theta_{11}} Z Y^{\theta_{13}}, \\ W_0 &\equiv \theta_{02} e^{\theta_{01}} Z Y^{\theta_{03}} \\ &= \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} \exp((\theta_{01} - \theta_{11}\theta_{03}/\theta_{13})Z) W_1^{\theta_{03}/\theta_{13}}. \end{aligned}$$

so that

$$\begin{aligned}
E_{\theta_1} W_1 &= 1 \\
E_{\theta_1} W_0 &= E_{\theta_1} (W_1^{\theta_{03}/\theta_{13}}) \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} E_{\theta_1} \exp((\theta_{01} - \theta_{11}\theta_{03}/\theta_{13})Z) \\
&= \Gamma(\theta_{03}/\theta_{13} + 1) \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} \{\theta_{14} \exp(\theta_{01} - \theta_{11}\theta_{03}/\theta_{13}) + (1 - \theta_{14})\}.
\end{aligned}$$

It remains only to compute

$$\begin{aligned}
E_{\theta_1} \log Y &= \frac{1}{\theta_{13}} E_{\theta_1} \log Y^{\theta_{13}} \\
&= \frac{1}{\theta_{12}\theta_{13}} E_{\theta_1} \{\exp(-\theta_{11}Z) E_{\theta_1} \{\theta_{12} \exp(\theta_{11}Z) \log Y^{\theta_{13}} | Z\}\} \\
&= \frac{1}{\theta_{12}\theta_{13}} E_{\theta_1} \{\exp(-\theta_{11}Z)\} E_{\theta_1}(W_1) \\
&= \frac{1}{\theta_{12}\theta_{13}} \{\theta_{14} \exp(-\theta_{11}) + (1 - \theta_{14})\} \psi(1)
\end{aligned}$$

where  $\psi(1) = -\gamma$  and  $\gamma$  is Euler's constant.

3. Consider the Weibull family of example 3.2.5:  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  with  $\Theta \subset R^{+2}$  given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right) 1_{[0,\infty)}(x)$$

where  $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$ . Suppose that  $X, X_1, \dots, X_n$  are i.i.d. with density function  $p_\theta$ .

A. If  $X \sim P_\theta \in \mathcal{P}$ , show that the distributions of  $\log X$  form a location and scale family from a Gumbel (extreme value) density on  $R$ .

B. Use the result of A to construct method of moments estimators or quantile based estimators  $\bar{\theta}_n$  of  $\theta = (\alpha, \beta)$ .

C. Show that the method of moments or quantile estimators  $\bar{\theta}_n$  of  $\theta$  are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\bar{\theta}_n - \theta) \rightarrow_d N_2(0, \Sigma) \quad \text{for some } \Sigma.$$

D. Does a maximum likelihood estimate of  $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$  exist? Is it unique?

E. Compute an approximate (one - step) maximum likelihood estimate  $\check{\theta}$  of  $\theta$  using the method of moment estimators  $\bar{\theta}_n$  as the preliminary estimators based on the following data (with  $n = 19$ ):

$$\begin{aligned}
&0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, \\
&7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.
\end{aligned}$$

[These are failure times in minutes for an insulating fluid between two electrodes subject to a voltage of 34 kV. – from Nelson, *Applied Life Data Analysis*, page 105.]

F. Compute the maximum likelihood estimator  $\hat{\theta}_n$ , and compare it with the one step estimator computed in E.

**Solution:** A. Recall that  $Y \equiv (X/\alpha)^\beta \sim \exp(1)$ , and that  $W \equiv -\log(Y) \sim \text{Gumbel}$ :

$$P(W \leq w) = P(-\log(Y) \leq w) = P(Y \geq e^{-w}) = \exp(-e^{-w}).$$

Thus it follows that

$$W = -\log(Y) = \beta\{-\log(X) + \log(\alpha)\},$$

or equivalently that

$$T \equiv -\log(X) = \frac{1}{\beta}W - \log(\alpha).$$

Thus the distributions of  $T \equiv -\log(X)$  form a location - scale family of the Gumbel (extreme value) distribution with d.f.  $\exp(-\exp(-x))$ .

B. Now  $T = -\log X$  has

$$E(T) = \frac{\gamma}{\beta} - \log \alpha, \quad \text{Var}(T) = \frac{1}{\beta^2} \frac{\pi^2}{6}$$

where  $\gamma = .577\dots$  is Euler's constant (don't confuse this with the  $\gamma$  above!). Since  $\bar{T} = -1.7864\dots$  and  $S_T = 1.4853\dots$ , moment estimators of  $(\alpha, \beta)$  based on (8) are given by

$$\begin{aligned} \bar{\beta}_n &\equiv \frac{\pi}{\sqrt{6}} \frac{1}{S_T} = .8639, \\ \bar{\alpha} &= \exp(-\bar{T} + \frac{\gamma}{\beta}) = 11.6407 \end{aligned}$$

for the given data.

C. Asymptotic normality of  $(\bar{\alpha}_n, \bar{\beta}_n)$  follows from joint asymptotic normality of  $(\bar{T}_n, S_T^2)$  and the delta method: by the multivariate CLT and Slutsky's theorem

$$\left( \begin{array}{c} \sqrt{n}(\bar{T} - ET)/\sigma \\ \sqrt{n}(S_T^2 - \sigma_T^2)/(\sqrt{2}\sigma_T^2) \end{array} \right) \rightarrow_d \underline{Z} \sim N_2(0, \Sigma).$$

Then since  $(\bar{\alpha}, \bar{\beta}) = g(\bar{T}, S_T^2)$  and  $(\alpha, \beta) = g(E_\theta T, \text{Var}_\theta(T))$  where  $g \equiv (g_1, g_2) : R^2 \rightarrow R^2$  is defined by

$$\begin{aligned} g_1(x, y) &= \exp\left(\frac{\gamma\sqrt{6}}{\pi}\sqrt{y} - x\right), \\ g_2(x, y) &= \frac{\pi/\sqrt{6}}{\sqrt{y}}, \end{aligned}$$

it follows by the delta method with  $\tilde{\underline{Z}} \equiv (Z_1, \sqrt{2}\sigma_T^2 Z_2)$  that

$$\sqrt{n}((\bar{\alpha}_n, \bar{\beta}_n)^T - (\alpha, \beta)^T) \rightarrow_d \nabla g \tilde{\underline{Z}}$$

where

$$\nabla g \equiv \nabla g(E_\theta T, \text{Var}_\theta T) = \left( \begin{array}{cc} -\alpha & (3\gamma/\pi^2)\alpha\beta \\ 0 & -3\beta^3/\pi^2 \end{array} \right).$$

D. The maximum likelihood estimator exists and is unique in this model if not all the  $X_i$ 's are equal (which happens with probability 1 if the model holds). The following solution is from Lehmann, TPE, page 536 (with slightly different notation).

We first reparametrize the Weibull model by writing

$$\begin{aligned} p_\theta(x) &= \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right) 1_{(0,\infty)}(x) \\ &= \frac{\beta}{\eta} x^{\beta-1} \exp\left(-\frac{x^\beta}{\eta}\right) \\ &\equiv p_\gamma(x) \end{aligned}$$

where  $\eta \equiv \alpha^\beta$  and  $\gamma \equiv (\beta, \eta)$ . Then

$$l(\gamma|\underline{X}) = n \log \beta - n \log \eta + (\beta - 1) \sum_{i=1}^n \log X_i - \frac{1}{\eta} \sum_{i=1}^n X_i^\beta.$$

Thus, with  $\gamma_1 \equiv \beta$ ,  $\gamma_2 \equiv \eta$ , the likelihood equations become

$$l_1(\gamma|\underline{X}) = \frac{n}{\beta} + \sum_{i=1}^n \log X_i - \frac{1}{\eta} \sum_{i=1}^n X_i^\beta \log X_i = 0, \quad (0.18)$$

and

$$l_2(\gamma|\underline{X}) = -\frac{n}{\eta} + \frac{1}{\eta^2} \sum_{i=1}^n X_i^\beta = 0, \quad (0.19)$$

or

$$\hat{\eta}_n = \frac{1}{n} \sum_{i=1}^n X_i^{\hat{\beta}} \quad (0.20)$$

from 0.19. Substitution of 0.20 into 0.18 yields the equation

$$\frac{\sum_i X_i^{\hat{\beta}} \log X_i}{\sum_i X_i^{\hat{\beta}}} - \frac{1}{\hat{\beta}} = \frac{1}{n} \sum_{i=1}^n \log X_i, \quad (0.21)$$

or

$$h(\hat{\beta}) = \frac{1}{n} \sum_{i=1}^n \log X_i \quad (0.22)$$

where

$$h(\beta) \equiv \frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta} - \frac{1}{\beta} < \frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta}$$

since  $\beta > 0$ . Now

$$\begin{aligned} h'(\beta) &= \frac{\sum_i X_i^\beta (\log X_i)^2}{\sum_i X_i^\beta} - \left(\frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta}\right)^2 + \frac{1}{\beta^2} \\ &\equiv I + II \\ &> I, \end{aligned}$$

and furthermore,

$$I = \sum a_i^2 p_i - \left(\sum a_i p_i\right)^2 = \text{Var}_p(a)$$

since, with  $a_i \equiv \log X_i$ ,  $p_i \equiv X_i^\beta / \sum_j X_j^\beta \geq 0$ ,  $\sum_i p_i = 1$ . Thus  $I > 0$  and hence  $h'(\beta) > 0$  from (0.23) while

$$-\infty = \lim_{\beta \rightarrow 0} h(\beta) < \frac{1}{n} \sum_{i=1}^n \log X_i < \log X_{(n)} = \lim_{\beta \rightarrow \infty} h(\beta).$$

[Draw the picture!] (To see this last limit, note that with  $p_{(i)} \equiv X_{(i)}^\beta / \sum_j X_j^\beta$ ,

$$\begin{aligned} p_{(i)} &= \frac{1}{\left(\frac{X_{(1)}}{X_{(i)}}\right)^\beta + \dots + \left(\frac{X_{(n)}}{X_{(i)}}\right)^\beta} \\ &\rightarrow \begin{cases} 0, & i \leq n \quad (\text{so } X_{(n)}/X_{(i)} > 1) \\ 1, & i = n \quad (\text{so } X_{(j)}/X_{(n)} < 1, j < n) \end{cases} \end{aligned}$$

as  $\beta \rightarrow \infty$ .) Thus (0.22) has a unique solution  $\hat{\beta}$ . By taking this value of  $\hat{\beta}$  in (0.20), we see that the MLE  $\hat{\gamma}$  of  $\gamma$  exists and is unique. Thus the unique MLE of  $\theta = (\alpha, \beta)$  is  $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$  with  $\hat{\alpha} = \hat{\eta}^{1/\hat{\beta}}$ .

E. The one step estimator using  $\hat{I}(\bar{\theta}_n) = I(\bar{\theta}_n)$  is

$$\check{\theta}_n \equiv \bar{\theta}_n + \hat{I}_n^{-1}(\bar{\theta}_n) \left( \frac{1}{n} \dot{l}(\bar{\theta}_n) \right) = (12.27 \dots, 0.7421 \dots).$$

The one - step estimator using  $\hat{I}_n(\bar{\theta}_n) = (-n^{-1} \ddot{l}_n(\bar{\theta}_n))$  is

$$\check{\theta}_n = (11.778, 0.7633).$$

F. The maximum likelihood estimate  $\hat{\theta}_n = (12.222\dots, 0.77082\dots)$ ; see the following pages.

### Mathematica input:

```
(* Here is the data: *)
x = {.19, .78, .96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85,
     6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91,
     36.71, 72.89}
(* NSS is the sample size *)
NSS:= Length[x]
(* Some useful functions: *)
(* f is the Weibull density function: *)
f[t_,a_,b_] := (b/a)*(t/a)^(b-1) *Exp[-(t/a)^b] ;

(* aa and bb are the constants in the Weibull Informaton: *)
aa := N[-(1-EulerGamma)];
bb := N[(Pi^2)/6 + aa^2 ]
(* Inf is the information matrix *)
Inf[a_,b_] := { {b^2/a^2 , aa/a}, {aa/a, bb/b^2} } ;
(* L is the log-likelihood *)
L[a_,b_] := Sum[Log[f[x[[i]], a,b]], {i,1,NSS} ] ;
(* Sc is the vector of Scores *)
```

```

Sc[a_,b_] := Sum[ {(b/a)((x[[i]]/a)^b -1),
(1/b)(1-Log[(x[[i]]/a)^b]*((x[[i]]/a)^b -1) ) },
{i,1,NSS}] ;
a[b_] := (Sum[x[[i]]^b, {i,1,NSS}]/NSS )^(1/b)

Plot3D[L[a,b], {a,4,15}, {b,.1,1.5}]
FindMinimum[-L[a,b], {a,10}, {b,2}]
ahat = 12.222
bhat = .770821
Wald[a_,b_] := NSS*({ahat,bhat} -{a,b}).Inf[ ahat,bhat].({ahat,bhat} -{a,b})
Rao[a_,b_] := Sc[a,b].Inverse[Inf[a,b]].Sc[a,b]/NSS
LR[a_,b_] := 2*(L[ahat,bhat] - L[a,b])
Wald[10,1]
Rao[10,1]
LR[10,1]

```

Mathematica output:

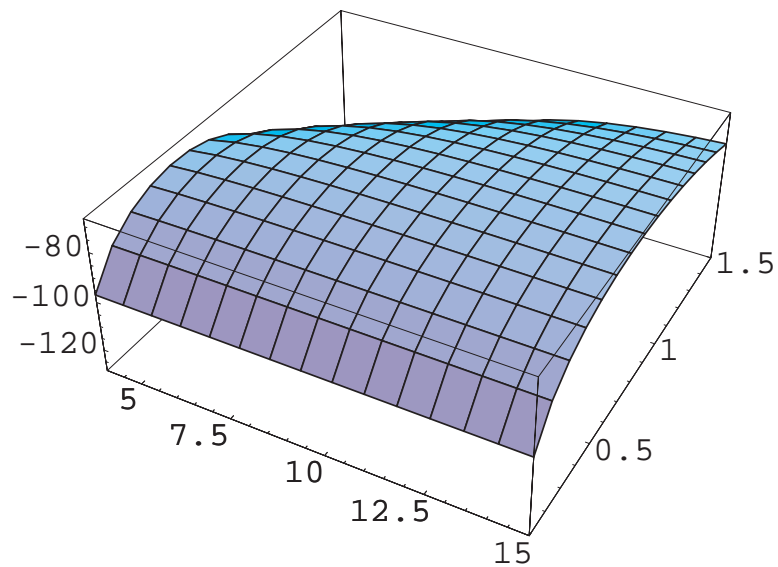


Figure 2: Weibull likelihood.

```

{68.386, {a -> 12.2222, b -> 0.770821}}
12.222
0.770821
4.10551
10.8385
5.29018

```