Statistics 581, Problem Set 1 Solutions

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1. Let X and Y be i.i.d. Uniform$(0,1)$ random variables Define $U = X - Y$, $V = \max(X,Y) = X \vee Y$.

   (i) What is the range of (U,V)?

   (ii) Find the joint density function $f_{U,V}(u,v)$ of the pair (U,V). Are U and V independent?

Solution: (i) The range of (X,Y) is
$A = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. The range of (U,V) is

$$B = \{(u,v) : 0 \le u \le 1, u \le v \le 1\} \cup \{(u,v) : -1 \le u < 0, -u \le v \le 1\}.$$

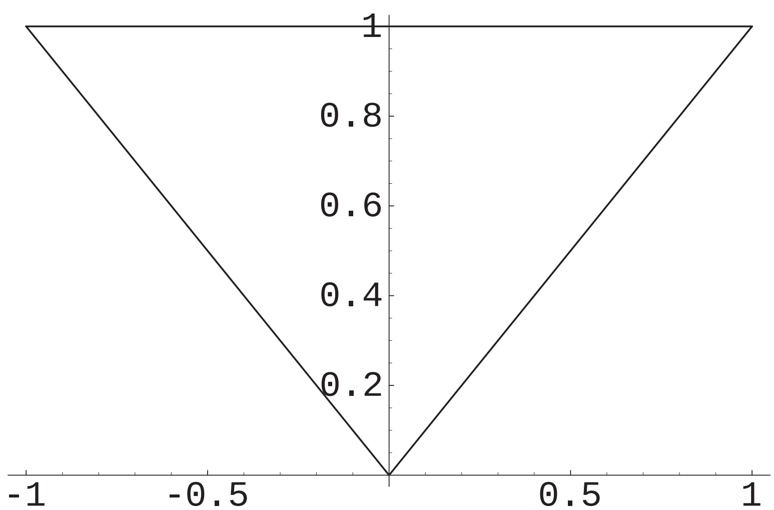

Figure 1: Range of U,V .

(ii) First solution - via Jacobians: The transformation $(X,Y) \to (U,V)$ is 1-1 and onto from A to B. On the set $x < y$, its inverse is given by $X = U + V$, $Y = V$; on the set $x > y$, its inverse is given by $X = V$, $Y = V - U$. These mappings are continuously differentiable on $B^* \equiv B \setminus \{(u, v) : (0, v)\} = B \setminus$ a null set. On B^* the Jacobian of the transformations are

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{if } x < y, \qquad \det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = 1 \qquad \text{if } x > y. \tag{1}$$

Thus by the usual transformation of densities formula, the joint density of (U, V) is obtained from $f_{X,Y}(x, y) = 1_{[0,1]}(x)1_{[0,1]}(y)$ as follows:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x(u, v), y(u, v))|\det \frac{\partial(x, y)}{\partial(u, v)}|1_{[x(u,v)<y(u,v)]} \\ &\qquad + f_{X,Y}(x(u, v), y(u, v))|\det \frac{\partial(x, y)}{\partial(u, v)}|1_{[x(u,v)>y(u,v)]} \\ &= \left(1_{[0,1]}(u + v)1_{[0,1]}(v)1_{[u+v<v]} + 1_{[0,1]}(v)1_{[0,1]}(v - u)1_{[v>v-u]}\right) \\ &= 1_B(u, v)\,. \end{aligned}$$

Thus the joint density of (U, V) is uniform on B. The random variables U and V are clearly *not* independent since the range of (U, V) is not a product set in R^2; moreover, the joint density of (U, V) does not factor into the product of its marginal densities. [The marginal densities are given by

$$f_U(u) = \int f_{U,V}(u, v)dv = \begin{cases} \int_u^1 dv = 1 - u, & u \in [0, 1] \\ \int_{-u}^1 dv = 1 + u, & u \in [-1, 0) \end{cases}$$

and

$$f_V(v) = \int f_{U,V}(u, v)du = \int_{-v}^{v} du = 2v1_{[0,1]}(v).]$$

Second solution by direction calculation of the joint distribution function: Note that we can write

$$\begin{aligned} &P(U \le u, V \le v) \\ &\quad = P(X - Y \le u, X \vee Y \le v) = P(X - Y \le u, X \le v, Y \le v) \\ &\quad = P(Y \ge X - u, X \le v, Y \le v) \\ &\quad = \begin{cases} v^2 - \frac{1}{2}(v - u)^2, & \text{if } \; 0 \le u \le v \le 1, \\ \frac{1}{2}(v + u)^2, & \text{if } \; -1 \le u < 0,\ 0 < -u \le v \le 1\,. \end{cases} \end{aligned}$$

(This is easy by pictures!) Computing $(\partial^2/\partial u \partial v)P(U \le u, V \le v)$ on each of these pieces separately again yields $f_{U,V}(u,v) = 1_B(u,v)$. Also note that the marginal distribution functions of U and V are given by $F_U(u) = (1/2)(1+u)^2 1_{[-1,0)}(u) + \{1 - \frac{1}{2}(1-u)^2\} 1_{[0,1]}(u)$ on $-1 \le u \le 1$ and $F_V(v) = v^2$ for $0 \le v \le 1$.

2. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Propostion 2.11, page 10.)
Solution. See Ferguson, ACILST page 173.

3. Suppose that for $\theta \in R$,

$$f_\theta(u,v) = \{1 + \theta(1-2u)(1-2v)\} 1_{[0,1]^2}(u,v).$$

(a) For what values of θ is f_θ a density function on $[0,1]^2$?
(b) For the set of θ's you identified in (a), find the corresponding distributon function F_θ and show that it has Uniform$(0,1)$ marginal distributions.
(c) If $(U,V) \sim F_\theta$, compute the correlation $\rho(U,V) \equiv \rho$ as a function of θ. Does this show any difficulty with this family of distributions as a model of dependence?
Solution: (a) For f_θ to be a density function, we must have $f_\theta(u,v) \ge 0$ for all $(u,v) \in [0,1]^2$ and

$$\int_0^1 \int_0^1 f_\theta(u,v)\,dudv = 1\,. \tag{2}$$

Now

$$\int_0^1 \int_0^1 f_\theta(u,v)\,dudv = 1 + \theta \int_0^1 \int_0^1 (1-2u)(1-2v)\,dudv = 1$$

for all $\theta \in R$ since

$$\int_0^1 \int_0^1 (1-2u)(1-2v)\,dudv = \int_0^1 (1-2u)\,du \int_0^1 (1-2v)\,dv = 0 \cdot 0 = 0\,,$$

and hence (2) holds for all θ. The requirement that f_θ be non-negative is just

$$1 + \theta(1-2u)(1-2v) \ge 0 \qquad \text{for all } \ (u,v) \in [0,1]^2\,,$$

or equivalently that

$$\theta(1-2u)(1-2v) \geq -1 \qquad \text{for all} \ \ (u,v) \in [0,1]^2 .$$

By monotonicity of $1-2u$, this holds if and only if it holds for $(u,v) \in \{(0,0),(0,1),(1,0),(1,1)\}$; i.e.

$$\theta \geq -1, \qquad -\theta \geq -1, \qquad -\theta \geq -1, \qquad \text{and} \ \ \theta \geq -1 .$$

Thus it follows that f_θ is a density function for $\theta \in [-1,1]$, or $|\theta| \leq 1$.
(b) The corresponding distribution function F_θ is given by

$$\begin{aligned}
F_\theta(u,v) &= \int_0^u \int_0^v f_\theta(r,s)drds \\
&= \int_0^u \int_0^v \{1+\theta(1-2r)(1-2s)\}drds \\
&= uv + \theta \int_0^u (1-2r)dr \int_0^v (1-2s)ds \\
&= uv + \theta u(1-u)v(1-v) \\
&= uv\left\{1+\theta(1-u)(1-v)\right\} .
\end{aligned}$$

Note that

$$F_\theta(u,1) = u\,, \quad \text{and} \ \ F_\theta(1,v) = v\,,$$

so F_θ has Uniform$(0,1)$ marginal distributions.
(c) It follows from part (iv) of Proposition 1.4.1, page 20, Chapter 1, that (by taking $G(x)=x$, $H(x)=x$))

$$\begin{aligned}
Cov(U,V) &= \int_0^1 \int_0^1 \{F_\theta(u,v) - uv\}dudv \\
&= \int_0^1 \int_0^1 \theta u(1-u)v(1-v)dudv \\
&= \theta \left(\int_0^1 u(1-u)du \right)^2 \\
&= \frac{1}{36}\theta
\end{aligned}$$

since

$$\int_0^1 u(1-u)du = \frac{1}{2}u^2 - \frac{1}{3}u^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} .$$

Now since $Var(U) = Var(V) = 1/12$ (since they are both Uniform$(0,1)$), it follows that

$$\rho(U,V) = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{\theta/36}{\sqrt{(1/12)(1/12)}} = \frac{\theta}{3}\,.$$

Note that this implies that $|\rho(U,V)| \le 1/3$, and hence this family of distributions does not include any distributions on $[0,1]^2$ with correlations larger than $1/3$ in absolute value.

4. Suppose that F is the distribution function of random variables X, Y with $X \sim$ Uniform$(0,1)$ marginally and $Y \sim$ Uniform$(0,1)$ marginally. Thus $F(x,y) = P(X \le x, Y \le y)$ satisfies

$$F(x,1) = x, \quad 0 \le x \le 1, \qquad \text{and} \qquad F(1,y) = y, \quad 0 \le y \le 1.$$

(a) Show that

$$F(x,y) \le x \wedge y \equiv F_U(x,y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. Here $x \wedge y \equiv x$ if $x \le y$, y if $y \le x$.

(b) Show that

$$F(x,y) \ge (1-(x+y))^+ \equiv F_L(x,y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. Here $z^+ = z 1_{[0,\infty)}(z)$.

(c) Show that F_U is the distribution function of (X,X) where $X \sim$ Uniform$(0,1)$. Show that F_L is the distribution function of $(X, 1-X)$ where $X \sim$ Uniform$(0,1)$.

(d) The distribution functions F_U and F_L are called the Fréchet bounds. Show that F_L and F_U are singular with respect to Lebesgue measure λ_2 on $[0,1]^2$; i.e. show that the corresponding probability measures P_L and P_U satisfy

$$P((X,Y) \in A) = 1, \qquad \lambda_2(A) = 0$$

and

$$P((X,Y) \in A^c) = 0, \qquad \lambda_2(A^c) = 1$$

for some set A (which will be different for P_L and P_U). This implies that F_L and F_U do not have densities with respect to Lebesgue measure on $[0,1]^2$. (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

Solution: (a) By monotonicity of F in each argument,

$$F(x,y) \le F(x,1) = x \qquad \text{and} \qquad F(x,y) \le F(1,y)$$

for $0 \le x, y \le 1$. Hence it follows that

$$F(x,y) \le \min\{x,y\} \equiv F_U(x,y), \qquad 0 \le x, y \le 1.$$

(b) Note that

$$0 \le P(X > x, Y > y) = 1 - F(x,1) - F(1,y) + F(x,y) = 1 - x - y + F(x,y)\,,$$

and hence

$$F(x,y) \ge x + y - 1, \qquad 0 \le x, y \le 1\,.$$

Since $F(x,y) = P(X \le x, Y \le y) \ge 0$ trivially, it follows that

$$F(x,y) \ge (x+y-1) \vee 0 = (x+y-1)^+ \equiv F_L(x,y) \qquad 0 \le x, y \le 1\,.$$

(c) If $X \sim \text{Uniform}(0,1)$, the joint distribution function of the pair $(X,Y) = (X,X)$ is given by

$$P(X \le x, Y \le y) = P(X \le x, X \le y) = P(X \le x \wedge y) = x \wedge y$$

for $0 \le x, y \le 1$. Thus F_U is the distribution function of (X,X). Similarly, the joint distribution function of the pair $(X,Y) = (X, 1-X)$ is given by

$$\begin{aligned} P(X \le x, 1 - X \le y) &= P(X \le x, X \ge 1 - y) \\ &= \begin{cases} P(1-y \le X \le x) & \text{if } x \ge 1-y \\ 0 & \text{if } x < 1-y \end{cases} \\ &= \begin{cases} x - (1-y) & \text{if } x \ge 1-y \\ 0 & \text{if } x < 1-y \end{cases} \\ &= (x+y-1)^+\,. \end{aligned}$$

Thus F_L is the distribution function of $(X, 1-X)$.
(d) It is clear from part (c) that if we take $A = \{(x,x) : \ 0 \le x \le 1\} \subset [0,1]^2$, then $P_U((X,Y) \in A) = 1$ and $\lambda_2(A) = 0$. Similarly, if $B = \{(x, 1-x) : \ 0 \le x \le 1\} \subset [0,1]^2$, then $P_L((X,Y) \in B) = 1$ and $\lambda_2(B) = 0$. Thus both P_U and P_L are singular with respect

to Lebesgue measure λ_2 on $[0,1]^2$, and the hypothesis of the Radon-Nikodym theorem 0.3.1 is violated. Hence neither P_U nor P_L have densities with respect to Lebesgue measure λ_2. Also note that

$$\frac{\partial^2}{\partial x \partial y} F_U(x,y) = 0 \quad \text{for} \quad (x,y) \notin A\,,$$

while

$$\frac{\partial^2}{\partial x \partial y} F_L(x,y) = 0 \quad \text{for} \quad (x,y) \notin B\,.$$

[Note that the arguments in (a) and (b) extend to an arbitrary distribution function F on R^2 with marginal d.f.'s F_X and F_Y respectively:

$$F(x,y) \le F_X(x) \wedge F_Y(y)\,,$$

and

$$F(x,y) \ge (F_X(x) + F_Y(y) - 1)^+\,.]$$

5. Lehmann and Casella, TPE, problem 1.10, page 62. Show that for real numbers $x_1, \ldots, x_n$ in the domain of a monotone real-valued functions h, the function

$$H(a) \equiv \frac{1}{n} \sum_{i=1}^{n} (h(x_i) - h(a))^2$$

is minimized by that value of a, say a_{min}, given by

$$a_{min} = h^{-1}\left(\frac{1}{n} \sum_{1}^{n} h(x_i) \right)\,.$$

Find particular functions h so that this yields the arithmetic, harmonic, and geometric means of the x_i's.
Solution: First note that that $H(a)$ is minimized by any value of a, say a_{min}, satisfying

$$h(a_{min}) = \frac{1}{n} \sum_{1}^{n} h(x_i)\,.$$

This follows by noting that

$$H(a) = H(a_{min}) + (h(a_{min}) - h(a))^2\,.$$

Since we have assumned that h is (strictly) monotone, it follows easily that

$$a_{min} = h^{-1}\left(\frac{1}{n}\sum_1^n h(x_i)\right) .$$

When $h(x) = x$,

$$a_{min} = \frac{1}{n}\sum_1^n (x_i) = \text{the arithmetic mean} .$$

When $h(x) = 1/x$,

$$a_{min} = \frac{1}{\frac{1}{n}\sum_1^n \frac{1}{x_i}} = \text{the harmonic mean} .$$

When $h(x) = \log(x)$,

$$\begin{aligned} a_{min} &= \exp\left(\frac{1}{n}\sum_1^n \log(x_i)\right) = \exp\left(\frac{1}{n}\log(\prod_1^n x_i)\right) \\ &= \exp\left(\log\left((\prod_1^n x_i)^{1/n}\right)\right) = \left(\prod_1^n x_i\right)^{1/n} \\ &= \text{the geometric mean} . \end{aligned}$$