Statistics 581, Problem Set 8

Wellner; 11/20/2002

Reading: Chapter 3, Section 2;
Ferguson, ACLST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443;
begin reading Chapter 4 (to be handed out on Friday 11/22).
Due: Wednesday, November 27, 2002.

1. Suppose that $\theta = (\theta_1, \theta_2) \in \Theta \subset R^k$ where $\theta_1 \in R$ and $\theta_2 \in R^{k-1}$. Show that:
A. $l_1^* = \dot{l}_1 - I_{12} I_{22}^{-1} \dot{l}_2$ is orthogonal to $[\dot{l}_2] \equiv \{a' \dot{l}_2 : a \in R^{k-1}\}$ in $L_2(P_\theta)$.
B. $I_{11 \cdot 2} = \inf_{c \in R^{k-1}} E_\theta (\dot{l}_1 - c' \dot{l}_2)^2$ and that the minimum is achieved when $c' = I_{12} I_{22}^{-1}$. Thus
$$I_{11 \cdot 2} = E_\theta (\dot{l}_1 - I_{12} I_{22}^{-1} \dot{l}_2)^2 = E_\theta [(l_\theta^*)^2] \, .$$
C. Prove the formulas (16) and (17) on page 17 and of the Chapter 3 notes and interpret these formulas geometrically.

2. Suppose that $(Y|Z) \sim \text{Weibull}(\lambda^{-1} e^{-\gamma Z}, \beta)$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. Thus the conditional cumulative hazard function $\Lambda(t|z)$ is given by
$$\Lambda_{\gamma,\lambda,\beta}(t|z) = (\lambda e^{\gamma Z} t)^\beta = \lambda^\beta e^{\beta \gamma Z} t^\beta$$
and hence
$$\lambda_{\gamma,\lambda,\beta}(t|z) = \lambda^\beta e^{\beta\gamma Z} \beta t^{\beta - 1} \, .$$
(Recall that $\lambda(t) = f(t)/(1 - F(t))$ and
$$\Lambda(t) \equiv \int_0^t \lambda(s) ds = \int_0^t (1 - F(s))^{-1} dF(s) = -\log(1 - F(t))$$
if F is continuous.) Thus it makes sense to reparametrize by defining $\theta_1 \equiv \beta\gamma$ (this is the parameter of interest since it reflects the effect of the covariate Z), $\theta_2 \equiv \lambda^\beta$, and $\theta_3 \equiv \beta$. This yields
$$\lambda_\theta(t|z) = \theta_3 \theta_2 \exp(\theta_1 z) t^{\theta_3 - 1}$$
You may assume that
$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$
exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", θ_1 is a "regression parameter" which affects the intensity of the (conditionally) Exponential variable Y, and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 \equiv \eta$.
(a) Derive the joint density $p_\theta(y, z)$ of (Y, Z) for the re-parametrized model.
(b) Find the information matrix for θ. What does the structure of this matrix say about the effect of $\eta = \theta_4$ being known or unknown about the estimation of $\theta_1, \theta_2, \theta_3$?
(c) Find the information and information bound for θ_1 if the parameters θ_2 and θ_3

are known?
(d) What is the information bound for θ_1 if just θ_3 is known to be equal to 1?
(e) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_3 is known.
(f) Find the information $I_{11\cdot(2,3)}$ and information bound for θ_1 if the parameters θ_2 and θ_3 are unknown. (Here both θ_2 and θ_3 are in "the second block".)
(g) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_2 and θ_3 are unknown.
(h) Specialize the calculations in (d) - (g) to the case when $Z \sim \text{Bernoulli}(\theta_4)$ and compare the information bounds.

3. **Optional bonus problem:** Information for location-scale families. Example 6.5, TPE page 126.
A. Confirm that Lehmann's information matrix for (regular) location-scale families is correct.
B. Verify that the off-diagonal term $I_{12} = 0$ when the location -scale family is from a density f that is symmetric about 0, and interpret this geometrically in terms of the scores for location $\mu = \theta_1$ and for scale $\sigma = \theta_2$.
C. Produce an example of a location-scale family which is not symmetric about 0 and hence for which $I_{12} \neq 0$. Compute the informatio matrix $I(\theta)$ as explicitly as possible in this case.

4. **Optional bonus problem:** Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta .$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $F_\theta(x) \equiv P_\theta(X \leq x)$.
(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page ??
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.