Statistics 581, Problem Set 4

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Reading: Course Notes, Chapter 2, sections 3-6; Ferguson, ACILST pages 44 - 66.
Due: Wednesday, October 30, 2002.

1. Suppose that $Y_i = \alpha + \theta'(x_i - \overline{x}) + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i \sim (0, \sigma^2)$ are i.i.d. and the x_i's are known vectors in R^k. Equivalently, $\underline{Y} = X\underline{\beta} + \underline{\epsilon}$ where

$$X^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 - \overline{x} & x_2 - \overline{x} & \cdots & x_n - \overline{x} \end{pmatrix}$$

so that X is an $n \times (k+1)$ matrix. Let $\hat{\underline{\beta}}$ be the least squares estimator of $\underline{\beta} = (\alpha, \theta')'$; i.e. $\hat{\underline{\beta}} = (X^T X)^{-1} X^T \underline{Y}$. Suppose that $n^{-1}(X^T X) \rightarrow D$ where D is positive definite.
(a) What additional condition(s) do you need to impose to prove that

$$\sqrt{n}(\hat{\beta}_n - \beta) \rightarrow_d N_{k+1}(0, \text{“}\textit{something}\text{”})\,?$$

(b) Find “something” in part (a).

2. Suppose that $X_1, X_2, \ldots$ are i.i.d. positive random variables, and define $\overline{X}_n \equiv n^{-1}\sum_{i=1}^n X_i$, $H_n \equiv 1/(n^{-1}\sum_{i=1}^n (1/X_i))$, and $G_n \equiv \{\prod_{i=1}^n X_i\}^{1/n}$ to be the *arithmetic, harmonic, and geometric* means respectively. We know that $\overline{X}_n \rightarrow_{a.s.} E(X_1) = \mu$ if and only if $E|X_1| < \infty$.
(a) Use the SLLN together with appropriate additional hypotheses to show that $H_n \rightarrow_{a.s.} 1/\{E(1/X_1)\} \equiv h$, and $G_n \rightarrow_{a.s.} \exp(E\{\log X_1\}) \equiv g$.
(c) Use the multivariate CLT and the delta method to find the joint limiting distribution of $\sqrt{n}(\overline{X}_n - \mu, H_n - h, G_n - g)$. You will need to impose or assume additional moment conditions to be able to prove this. Specify these additional assumptions carefully.
(d) Suppose that $X_i \sim$ Gamma(r, λ) with $r > 0$. For what values of r are the hypotheses you imposed in (c) satisfied? Compute the covariance of the limiting distribution in (c) as explicitly as you can in this case.
(e) Use the result in (c) to show that $\sqrt{n}(G_n/\overline{X}_n - g/\mu) \rightarrow_d N(0, V^2)$ and compute V^2 explicitly when $X_i \sim$ Gamma(r, λ) with r satisfying the conditions you found in (d).

3. Suppose that $\underline{N}_n = (N_{11}, N_{12}, N_{21}, N_{22}) \sim \text{Mult}_4(n, \underline{p})$ where $\underline{p} = (p_{11}, p_{12}, p_{21}, p_{22})$ where $\sum_{i=1}^2 \sum_{j=1}^2 p_{ij} = 1$. (Thus $\underline{N}_n$ is the sum of n independent $\text{Mult}_4(1, \underline{p})$ random vectors $\{\underline{Y}_i\}_{i=1}^n$.) Since there are really just three independently varying parameters for this problem, it is often useful to re-express the cell probabilities in terms of two marginal probabilities, say $p_{1\cdot} = p_{11} + p_{12}$ and $p_{\cdot 1} = p_{11} + p_{21}$, and ψ, the log of the odds-ratio, defined by

$$\psi \equiv \log \frac{p_{21}/p_{22}}{p_{11}/p_{12}} = \log \frac{p_{12}p_{21}}{p_{11}p_{22}}\,. \tag{1}$$

You may use the fact that $\psi = 0$ if and only if independence holds for the 2×2 table (i.e. $p_{ij} = p_{i\cdot}p_{\cdot j}$ for $i, j = 1, 2$).

(a) Suggest an estimator of ψ, say $\widehat{\psi}$.
(b) Show that the estimator you proposed in (a) is asymptotically normal and compute the asymptotic variance of your estimator.

4. This is a continuation of problem 3. One standard test of independence in the 2×2 table is the test based on a Pearson-type chi-square statistic.
   (a) Write down the chi-square statistic Q_n for this problem, state its asymptotic distribution under the null hypothesis, and explain briefly why the claimed result holds.
   (b) Suppose that the alternative hypothesis holds. Show that under the alternative hypothesis $n^{-1}Q_n \rightarrow_p$ some constant q and compute q as explicitly as possible.
   (c) Find the asymptotic distribution of Q_n under local alternatives of the form $\psi_n = tn^{-1/2}$; i.e. $\underline{p}_n \equiv (p_{11,n}, p_{12,n}, p_{21,n}, p_{22,n}) = \underline{p}_0 + \underline{c}n^{-1/2}$ where
$$\psi_0 \equiv \log\left(\frac{p_{21,0}p_{12,0}}{p_{11,0}p_{22,0}}\right) = 0$$
   and $\underline{1}'\underline{c} = 0$.
   (d) Suppose that $n = 30$, $\alpha = .02$, and the true $\underline{p}$ is $\underline{p} = (.3, .2, .1, .4)$. Give an approximation to the power of the chi-square test at this particular alternative.

5. **Optional bonus problem:** Ferguson, ACILST, problem 1, page 65.

6. **Optional bonus problem:** Ferguson, ACILST, problem 3, page 42.